

CHEN-RUAN COHOMOLOGY OF *ADE* SINGULARITIES

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Abstract

We study Ruan's *cohomological crepant resolution conjecture* [41] for orbifolds with transversal *ADE* singularities. In the A_n -case we compute both the Chen-Ruan cohomology ring $H_{\text{CR}}^*([Y])$ and the quantum corrected cohomology ring $H^*(Z)(q_1, \dots, q_n)$. The former is achieved in general, the later up to some additional, technical assumptions. We construct an explicit isomorphism between $H_{\text{CR}}^*([Y])$ and $H^*(Z)(-1)$ in the A_1 -case, verifying Ruan's conjecture. In the A_n -case, the family $H^*(Z)(q_1, \dots, q_n)$ is not defined for $q_1 = \dots = q_n = -1$. This implies that the conjecture should be slightly modified. We propose a new conjecture in the A_n -case (Conj. 1.9). Finally, we prove Conj. 1.9 in the A_2 -case by constructing an explicit isomorphism.

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0 Introduction

The Chen-Ruan cohomology was defined by Chen and Ruan [11] for almost complex orbifolds. This was extended to a non-commutative ring by Fantechi and Göttsche [18] in the case where the orbifold is a global quotient. Abramovich, Graber and Vistoli defined the Chen-Ruan cohomology in the algebraic case [1].

Let $[Y]$ be a complex Gorenstein orbifold such that the coarse moduli space Y admits a crepant resolution $\rho: Z \rightarrow Y$. Then, under some technical assumptions on Z , Ruan's *cohomological crepant resolution conjecture* [41] predicts the existence of an isomorphism between the Chen-Ruan cohomology ring $H_{\text{CR}}^*([Y], \mathbb{C})$ and the so called quantum corrected cohomology ring of Z . The later is a deformation of the ring $H^*(Z, \mathbb{C})$ obtained using certain Gromov-Witten invariants of rational curves in Z which are contracted under the resolution map ρ . Notice that if Z carries an holomorphic symplectic structure, then this conjecture also predicts the existence of an isomorphism between the Chen-Ruan cohomology ring of $[Y]$ and the cohomology ring of Z .

An interesting testing case for the conjecture is the one of the Hilbert scheme $\text{Hilb}^r M$ of r points on a projective surface M . It is a crepant resolution of the symmetric product $\text{Sym}^r M$ via the Chow morphism. In this case the conjecture was proved by W.-P. Li and Z. Qin for $r = 2$ [28], for r general and M with numerically trivial canonical class by Fantechi and Göttsche [18] (using the explicit computation of the ring $H^*(\text{Hilb}^r M)$ given by Lehn and Sorger [26]), and independently by Uribe [46]. A different and self-contained proof of this result was given by Z. Qin and W. Wang [37]. In the same situation but with M quasi-projective with a holomorphic symplectic form, the conjecture was proved by W.-P. Li, Z. Qin and W. Wang [29]. In particular this result generalizes the case of the affine plane obtained by Lehn and Sorger [27] and Vasserot [47] independently. The general case where $Y = V/G$ with V complex symplectic vector space and $G \subset Sp(V)$ finite subgroup was proved by Ginzburg and Kaledin [21]. Let us point out that in the previous cases (except [28]) the resolution Z carries a holomorphic symplectic structure, hence the quantum corrected cohomology ring coincides with the cohomology ring $H^*(Z, \mathbb{C})$. D. Edidin, W.-P. Li and Z. Qin partially verified Ruan's conjecture in the case where $M = \mathbb{P}^2$ and $r = 3$, there quantum corrections appeared [17].

The aim of this paper is to study Ruan's conjecture for orbifolds with transversal *ADE* singularities (see Def. 2.10). An orbifold $[Y]$ has transversal *ADE* singularities if, étale locally, the coarse moduli space Y is isomorphic to a product $R \times \mathbb{C}^k$, where R is a germ of an *ADE* singularity. Notice that for any Gorenstein orbifold $[Y]$, there exists a closed subset $W \subset Y$ of codimension ≥ 3 such

that $Y \setminus W$ has transversal ADE singularities. Thus the case we study is the general one if we ignore phenomena that occur in codimension ≥ 3 .

We describe the twisted sectors of orbifolds with ADE singularities. After that, we concentrate on the transversal A_n -case and we address Ruan's conjecture by computing explicitly both the Chen-Ruan cohomology (Th. 3.12) and the quantum corrections (Prop. 5.4). The former is achieved in general, regarding the later we propose a conjecture on the value of some Gromov-Witten invariants (Conj. 5.1) which is proved fully in the A_1 -case, and in the A_n -case ($n \geq 2$) under additional technical assumptions. In a work in progress with B. Fantechi we give a proof of Conj. 5.1 and compute the quantum corrections in the transversal D and E cases.

We construct an explicit isomorphism between the Chen-Ruan cohomology ring $H_{\text{CR}}^*([Y])$ and the quantum corrected cohomology ring $H^*(Z)(-1)$ in the transversal A_1 -case, verifying Ruan's conjecture (Sec. 6.1). In the A_n -case, the quantum corrected 3-point function can not be evaluated in $q_1 = \dots = q_n = -1$. This implies that Ruan's conjecture has to be slightly modified. We propose a modification in the A_n -case (Conj. 1.9) that we prove in the A_2 -case, by constructing an explicit isomorphism (see Prop. 6.2).

The structure of the paper is the following. In Section 1 we review the statement of the cohomological crepant resolution conjecture. Orbifolds with transversal ADE singularities are defined in Section 2. Then in Section 3, we compute explicitly the Chen-Ruan cohomology ring of such orbifolds. In Section 4 we prove that up to isomorphism the coarse moduli space of an orbifold with transversal ADE singularities has a unique crepant resolution Z and we describe the cohomology ring of Z . In Section 5, we state our conjecture about the Gromov-Witten invariants of Z (whose proof in some particular cases is postponed to Section 7). Using this, we compute the quantum corrected cohomology ring. Afterwards we put together these results to verify our modification of Ruan's conjecture.

Notation

We will work over the field of complex numbers \mathbb{C} . Through out this paper, Y and Z will denote projective algebraic varieties of dimension d over \mathbb{C} . The singular locus of Y is denoted by S and the inclusion by $i : S \rightarrow Y$.

A complex orbifold $[Y]$ means a complex orbifold structure over the topological space Y . In this context, Y has the complex topology. Our references for orbifolds are [10], [11], [33] and [36]. In particular notations are taken from [10] and [36].

We will work with cohomology groups with complex coefficients, although many results are valid for rational coefficients.

1 The cohomological crepant resolution conjecture

In this section we recall the statement of the *cohomological crepant resolution conjecture* as given by Y. Ruan in [41]. The conjecture claims a precise relation between the Chen-Ruan cohomology ring of a complex orbifold $[Y]$ and the cohomology ring of a crepant resolution of Y , when such a resolution exists.

Definition 1.1. *A complex orbifold $[Y]$ is Gorenstein if the degree shifting numbers $\iota_{(g)}$ are integers, for all $(g) \in T$.*

Notice that, if $[Y]$ is Gorenstein, then the algebraic variety Y is also Gorenstein and in particular the canonical sheaf K_Y is locally free (see e.g. [38] and [39] for more details).

Definition 1.2 ([39]). *Let Y be a Gorenstein variety. A resolution of singularities $\rho : Z \rightarrow Y$ is crepant if $\rho^*(K_Y) \cong K_Z$.*

Crepant resolutions of Gorenstein varieties with quotient singularities are known to exist in dimensions 2 and 3. In particular, for $d = 2$ a stronger result holds: every normal surface Y admits a unique crepant resolution [2]. In dimension $d = 3$ the existence of a crepant resolution is proven e.g. in [40] and in [9], however the uniqueness result does not hold. In dimension $d \geq 4$ crepant resolutions not always exist.

We will work under the following

Assumption 1.3. Let $[Y]$ be a Gorenstein orbifold and $\rho : Z \rightarrow Y$ a fixed crepant resolution. Then consider the induced group homomorphism

$$\rho_* : H_2(Z, \mathbb{Q}) \rightarrow H_2(Y, \mathbb{Q}). \quad (1)$$

We assume that the extremal rays contracted by ρ are generated by n rational curves whose homology classes β_1, \dots, β_n are linearly independent over \mathbb{Q} . Then β_1, \dots, β_n determine a basis of $\text{Ker } \rho_*$ called *integral basis* [41].

The homology class of any effective curve that is contracted by ρ can be written in a unique way as $\Gamma = \sum_{l=1}^n a_l \beta_l$, with the a_l 's positive integers. For each β_l we assign a formal variable q_l , so Γ corresponds to $q_1^{a_1} \cdots q_n^{a_n}$. The *quantum corrected 3-point function* is

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, \dots, q_n) := \sum_{a_1, \dots, a_n > 0} \Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) q_1^{a_1} \cdots q_n^{a_n}, \quad (2)$$

where $\gamma_1, \gamma_2, \gamma_3 \in H^*(Z)$ are cohomology classes, $\Gamma = \sum_{l=1}^n a_l \beta_l$, and $\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3)$ is the genus zero Gromov-Witten invariant of Z [41].

Assumption 1.4. We assume that (2) defines an analytic function of the variables q_1, \dots, q_n on some region of the complex space \mathbb{C}^n . It will be denoted by $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}$. In the following, when we evaluate $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}$ on a point (q_1, \dots, q_n) , we will implicitly assume that it is defined on such a point.

We now define a family of rings depending on the parameters q_1, \dots, q_n .

Definition 1.5. The *quantum corrected triple intersection* $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, \dots, q_n)$ is defined by

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\rho(q_1, \dots, q_n) := \langle \gamma_1, \gamma_2, \gamma_3 \rangle + \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, \dots, q_n),$$

where $\langle \gamma_1, \gamma_2, \gamma_3 \rangle := \int_Z \gamma_1 \cup \gamma_2 \cup \gamma_3$. The *quantum corrected cup product* $\gamma_1 *_\rho \gamma_2$ is defined by requiring that

$$\langle \gamma_1 *_\rho \gamma_2, \gamma \rangle = \langle \gamma_1, \gamma_2, \gamma \rangle_\rho(q_1, \dots, q_n) \quad \text{for all } \gamma \in H^*(Z),$$

where $\langle \gamma_1, \gamma_2 \rangle := \int_Z \gamma_1 \cup \gamma_2$.

Remark 1.6. Our definition of quantum corrected triple intersection and of quantum corrected cup product is slightly different from the one given in [41]. One can recover the original definition by giving to the parameters the value $q_1 = \dots = q_n = -1$, provided that this point belongs to the domain of the quantum corrected 3-point function.

Proposition 1.7 ([13]). *For any (q_1, \dots, q_n) belonging to the domain of the quantum corrected 3-point function, the quantum corrected cup product $*_\rho$ satisfies the following properties.*

Associativity: it is associative on $H^(Z)$, moreover it has a unit which coincides with the unit of the usual cup product of Z .*

*Skewsymmetry: $\gamma_1 *_\rho \gamma_2 = (-1)^{\deg \gamma_1 \cdot \deg \gamma_2} \gamma_2 *_\rho \gamma_1$, for any $\gamma_1, \gamma_2 \in H^*(Z)$.*

Homogeneity: for any $\gamma_1, \gamma_2 \in H^(Z)$, $\deg(\gamma_1 *_\rho \gamma_2) = \deg \gamma_1 + \deg \gamma_2$.*

Definition 1.8. The quantum corrected cohomology ring of Z is the family of ring structures on the vector space $H^*(Z)$ given by $*_\rho$. It will be denoted by $H_\rho^*(Z)(q_1, \dots, q_n)$.

We finally come to Ruan's conjecture, whose study is the reason of this paper.

Cohomological crepant resolution conjecture (Y. Ruan, [41])

Under the above hypothesis, there exists a ring isomorphism

$$H_\rho^*(Z)(-1, \dots, -1) \cong H_{\text{CR}}^*([Y]).$$

As said, this conjecture needs to be slightly modified. In the A_n -case we propose the following

Conjecture 1.9. *Let $[Y]$ be an orbifold with transversal A_n -singularities and trivial monodromy (Def. 3.4), $\rho: Z \rightarrow Y$ be the crepant resolution (Prop. 4.2). Then the following map*

$$H_\rho^*(Z)(q_1, \dots, q_n) \cong H_{\text{CR}}^*([Y]) \quad (3)$$

$$E_l \mapsto \sum_{k=1}^n \zeta^{lk} (\zeta^k + \zeta^{-k} - 2)^{1/2} e_k$$

is a ring isomorphism for $q_1 = \dots = q_n = \zeta$ be a primitive $(n+1)$ -th root of 1. Here E_1, \dots, E_n are the irreducible components of the exceptional divisor (see Notation 4.5) and e_1, \dots, e_n are the generators of the Chen-Ruan cohomology (see Thm. 3.12). The square root in (3) means, for $\zeta = \exp\left(\frac{2\pi i m}{n+1}\right)$,

$$(\zeta^k + \zeta^{-k} - 2)^{1/2} = \begin{cases} i|(2 - \zeta^k - \zeta^{-k})^{1/2}| & \text{if } 0 < m < \frac{n+1}{2}; \\ -i|(2 - \zeta^k - \zeta^{-k})^{1/2}| & \text{otherwise.} \end{cases}$$

Remark 1.10. The isomorphism in the previous conjecture is the one conjectured by J. Bryan, T. Graber and R. Pandharipande [7] for the A_n -case. It coincides with the map found by W. Nahm and K. Wendland [34]. In a recent work, joint with S. Boissière and E. Mann [5], we prove that (3) gives an isomorphism between the Chen-Ruan cohomology ring of the weighted projective space $\mathbb{P}(1, 3, 4, 4)$ and the quantum corrected cohomology ring of its crepant resolution. We expect to report on the verification of Conj. 1.9 soon.

In Chapter 2.2 we will see how to get (3) from the classical McKay correspondence.

2 Orbifolds with ADE singularities

In this Section we define orbifolds with transversal ADE singularities. They are generalizations of Gorenstein orbifolds associated to quotient surface singularities, also called *rational double points*. Therefore we first recall the definition of such surface singularities and collect some properties. We will follow [2], [15], [16].

2.1 Rational double points

Definition 2.1. A *rational double point* (in short *RDP*) is the germ of a surface singularity $R \subset \mathbb{C}^3$ which is isomorphic to a quotient \mathbb{C}^2/G with G a finite subgroup of $SL(2, \mathbb{C})$.

Rational double points are Gorenstein. Indeed every variety with symplectic singularities is Gorenstein [3].

Finite subgroups of $SL(2, \mathbb{C})$ are classified, up to conjugation, and the result of this classification is given in the following Theorem.

Theorem 2.2 ([16]). *Any finite subgroup of $SL(2, \mathbb{C})$ is conjugate to one of the following subgroups: the binary tetrahedral group E_6 of order 24; the binary octahedral group E_7 of order 48; the binary icosahedral group E_8 of order 120; the binary dihedral group D_n of order $4(n-2)$ for $n \geq 4$; the cyclic group A_n of order $n+1$.*

It turns out that conjugate subgroups give isomorphic surface singularities. Hence the above classification induces a classification of RDP's [16]:

$$\begin{aligned} A_n : & \quad xy - z^{n+1} = 0 & \text{for } n \geq 1 \\ D_n : & \quad x^2 + y^2z + z^{n-1} = 0 & \text{for } n \geq 4 \\ E_6 : & \quad x^2 + y^3 + z^4 = 0 \\ E_7 : & \quad x^2 + y^3 + yz^3 = 0 \\ E_8 : & \quad x^2 + y^3 + z^5 = 0. \end{aligned} \tag{4}$$

Resolution graph

Any rational double point R has a unique crepant resolution $\rho : \tilde{R} \rightarrow R$ [2]. The exceptional locus of ρ is the union of rational curves E_1, \dots, E_n with self-intersection numbers -2 . Moreover, it is possible to associate a graph to the collection of these curves in the following way: there is a vertex for any irreducible component of the exceptional locus; two vertices are joined by an edge if and only if the corresponding components have non zero intersection. The list of the graphs obtained by resolving rational double points is given in [15] and in [2]. Each of this graph is called *resolution graph* of the corresponding singularity.

Notation 2.3. From now on, R will denote a surface in \mathbb{C}^3 defined by one of the equations (4), i.e. a surface with a rational double point at the origin $0 \in \mathbb{C}^3$. The crepant resolution of R will be denoted by $\rho : \tilde{R} \rightarrow R$.

2.2 McKay correspondence

Let R be a RDP and $G \subset SL(2, \mathbb{C})$ be a finite subgroup corresponding to R . We denote by $Q = \mathbb{C}^2$ the representation induced by the inclusion $G \subset SL(2, \mathbb{C})$. Let $\lambda_0, \dots, \lambda_m$ be the (isomorphism classes of) irreducible representations of G , with λ_0 being the trivial one. Then, for any $j = 1, \dots, m$ we can decompose $Q \otimes \lambda_j$ as follows

$$Q \otimes \lambda_j = \bigoplus_{i=0}^m a_{ij} \lambda_i, \quad a_{ij} = \dim_{\mathbb{C}} \text{Hom}_G(\lambda_i, Q \otimes \lambda_j). \tag{5}$$

Definition 2.4. The *McKay graph* of $G \subset SL(2, \mathbb{C})$ is the graph with one vertex for any irreducible representation, two vertices are joined by a_{ij} arrows. It will be denoted by $\tilde{\Gamma}_G$. If we consider only nontrivial representations, then we obtain the graph Γ_G , which will be called also *McKay graph*.

Remark 2.5. In [32] the *representation graph* of G (i.e. what we call the McKay graph) was defined in a slightly different way. However it can be shown that, for finite subgroups of $SL(2, \mathbb{C})$, the two definitions coincide.

The McKay correspondence, in his original form, states that the graph Γ_G coincides with the resolution graph of R . The correspondence can be obtained geometrically by means of a map that identifies the K-theory of the orbifold $[R]$ with that of \tilde{R} , this is done in [22]. We recall briefly this construction.

A G -equivariant coherent sheaf on \mathbb{C}^2 is a coherent sheaf F on \mathbb{C}^2 together with isomorphisms

$$\alpha_g : g^* F \rightarrow F, \quad g \in G$$

which satisfy the obvious cocycle condition. Let $K([R])$ the Grothendieck ring of isomorphism classes of G -equivariant coherent sheaves on \mathbb{C}^2 . As usual, $K(\tilde{R})$ denotes the Grothendieck ring of isomorphism classes of coherent sheaves on \tilde{R} . Finally, set $R(G)$ be the ring of isomorphism classes of representations of G . For any $\lambda \in R(G)$, λ^\vee denotes the dual class.

We have the following

Proposition 2.6 ([22]). *The map that associates, to any representation λ of G on the vector space V_λ , the G -equivariant coherent sheaf $\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda^\vee}$ induces a ring isomorphism*

$$R(G) \xrightarrow{\cong} K([R]).$$

We identify the two rings by means of this map.

Consider now the Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathbb{C}}^2 & \xrightarrow{\text{pr}_2} & \tilde{R} \\ \text{pr}_1 \downarrow & & \downarrow \rho \\ \mathbb{C}^2 & \xrightarrow{\chi} & R \end{array}$$

where χ is the quotient map. The following result holds.

Theorem 2.7 ([22]). *Let*

$$\pi : R(G) = K([R]) \rightarrow K(\tilde{R})$$

defined by

$$\pi := \text{Inv} \circ \text{pr}_{2*} \circ \text{pr}_1^*,$$

where pr_{2} and pr_1^* are the canonical morphisms and Inv is the application that associates to any G -equivariant coherent sheaf M on \tilde{R} the subsheaf M^G of the invariants. Then*

- (i) *for any irreducible representation λ of G , there is a unique component E_λ of the exceptional divisor E such that*

$$\text{rk}(\pi(\lambda)) = \deg \lambda \quad \text{and} \quad c_1(\pi(\lambda)) = c_1(\mathcal{O}_{\tilde{R}}(E_\lambda)).$$

The map $\lambda \mapsto E_\lambda$ is a bijection from the set of irreducible representations of G to the set of components of E . For any $\lambda \neq \mu$, $(E_\lambda \cdot E_\mu) = a_{\lambda\mu}$, where the $a_{\lambda\mu}$'s are defined in (5) and $(_ \cdot _)$ is the Poincaré pairing.

- (ii) *π is an isomorphism of \mathbb{Z} -modules.*

This Thm. can be used to get a correspondence between the Chen-Ruan cohomology of $[R]$ and the cohomology of \tilde{R} as follows (we refer to the next Chapter for the definition of Chen-Ruan cohomology). We have maps

$$\text{Ch}(_) \cdot \text{Td}(\tilde{R}) : K(\tilde{R}) \rightarrow H^*(\tilde{R}) \quad (6)$$

$$\text{Ch}(_) \cdot \text{Td}([R]) : K([R]) \rightarrow H_{\text{CR}}^*([R]) \quad (7)$$

where Ch and Td are the usual Chen character and Todd class respectively, Ch and Td are the Chern character and Todd class for orbifolds as defined by Toen [44], and the multiplications are the usual cup products (not the Chen-Ruan one in the second case). Then the map π of Thm.2.7, (6) and (7) give a map between cohomology groups. We work out the details of this computation in the A_n -case.

Identify the group G with \mathbb{Z}_{n+1} and set $\zeta = \exp(\frac{2\pi i}{n+1}) \in \mathbb{C}^*$. Let λ_m be the irreducible representation of \mathbb{Z}_{n+1} on V_{λ_m} whose character is

$$l \mapsto \zeta^{ml}.$$

From Thm. 2.7 we have that

$$\text{Ch}(\pi(\lambda_m)) \cdot \text{Td}(\tilde{R}) = 1 + c_1(\mathcal{O}_{\tilde{R}}(E_{\lambda_m})) \in H^*(\tilde{R}). \quad (8)$$

We compute now

$$\text{Ch}(\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda_m^\vee}) \cdot \text{Td}([R]) \in H_{\text{CR}}^*([R]). \quad (9)$$

For any $l \in \mathbb{Z}_{n+1}$, consider the restriction $(\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda_m^\vee})|_{(\mathbb{C}^2)^l}$ of $\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda_m^\vee}$ to the fixed point locus $(\mathbb{C}^2)^l$ of l . The action of l on $(\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda_m^\vee})|_{(\mathbb{C}^2)^l}$ is given by the multiplication by ζ^{-lm} . Hence

$$\text{Ch}(\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda_m^\vee}) = \sum_{l \in \mathbb{Z}_{n+1}} \zeta^{-lm} \cdot 1_{H^*(R_{(l)})},$$

where $1_{H^*(R_{(l)})}$ is the neutral element of the cohomology ring of the twisted sector $R_{(l)}$, for any $l \in \mathbb{Z}_{n+1}$. Next we compute the class $\alpha_{[R]} \in K([R_1])$ defined in [44], where $[R_1]$ is the inertia orbifold. We denote by C the conormal sheaf of $[R_1]$ with respect to $[R]$, i.e. the sheaf on $[R_1]$ whose restriction to each twisted sector is the conormal sheaf of the twisted sector in $[R]$. For any $l \in \mathbb{Z}_{n+1}$, set C_l the restriction of C to $[R_{(l)}]$. Then, if $l = 0$, C_l has rank 0. Otherwise it is given by the representation $\lambda_1 \oplus \lambda_n$ of \mathbb{Z}_{n+1} .

$$\lambda_{-1}(C) = 1 - C + \wedge^2 C,$$

hence

$$(\alpha_{[R]})|_{(\mathbb{C}^2)^l} = \begin{cases} 1 & \text{if } l = 0; \\ 2 - \zeta^l - \zeta^{-l} & \text{otherwise.} \end{cases}$$

Therefore

$$\text{Td}([R]) = 1_{H^*(R_{(0)})} + \sum_{l=1}^n \frac{1}{2 - \zeta^l - \zeta^{-l}} \cdot 1_{H^*(R_{(l)})}.$$

Finally, we get

$$\text{Ch}(\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} V_{\lambda_m}) \cdot \text{Td}([R]) = 1_{H^*(R_{(0)})} + \sum_{l=1}^n \frac{\zeta^{-lm}}{2 - \zeta^l - \zeta^{-l}} \cdot 1_{H^*(R_{(l)})}.$$

Remark 2.8. The previous procedure gives the following map

$$\begin{aligned} H^2(\tilde{R}) &\rightarrow H_{\text{CR}}^2([R]) \\ E_m &\mapsto \sum_{l=1}^n \frac{\zeta^{-lm}}{2 - \zeta^l - \zeta^{-l}} e_l, \end{aligned}$$

where we have used the same notation as in Conj. 1.9. It follows from Prop. 6.2 that this is not a ring isomorphism. But it is clear how to change the procedure to get the correct map.

However the previous computation gives a way to get the isomorphism between the Chen-Ruan cohomology and the quantum corrected cohomology of the crepant resolution in the *ADE*-case. This will be object of further investigations.

2.3 Definition of orbifolds with transversal *ADE* singularities

We use the language of groupoids, and refer to [10] and to the references there for a more detailed discussion of the relations between orbifolds and groupoids. To fix notations, we recall that an *orbifold structure* on the paracompact Hausdorff space Y is defined to be an orbifold groupoid \mathcal{G} with a homeomorphism $f : |\mathcal{G}| \rightarrow Y$. Two orbifold structures (\mathcal{G}, f) and (\mathcal{G}', f') are *equivalent* iff \mathcal{G} and \mathcal{G}' are Morita equivalent and the maps f and f' are compatible under the equivalence relation. Then an *orbifold* $[Y]$ is defined to be a space Y with an equivalent class of orbifold structures. An orbifold structure (\mathcal{G}, f) in such an equivalence class is a *presentation* of the orbifold $[Y]$. The orbifold $[Y]$ is *complex* if it is given in addition a complex structure on the tangent bundle TG_0 , which is equivariant under the \mathcal{G} -action.

An orbifold structure over Y can also be given by an open covering $\{V_\alpha\}$ of Y and, for any α , a smooth variety U_α , a finite group G_α acting on it, and an homeomorphism $\chi_\alpha : U_\alpha/G_\alpha \rightarrow V_\alpha$. This data must satisfies the condition that, whenever $u \in U_\alpha$ and $u' \in U_\beta$ map to the same $y \in Y$, then

there exist neighborhoods $W \subset U_\alpha$ of u and $W' \subset U_\beta$ of u' , and an isomorphism $\varphi : W \rightarrow W'$ which sends u in u' such that the following diagram commutes

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & W' \\ \chi_\alpha \downarrow & & \downarrow \chi_\beta \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

Then, if we set

$$G_0 := \sqcup_\alpha U_\alpha,$$

$G_1 := \{(u, \varphi, u') | u \text{ and } u' \text{ map to the same } y \in Y, \text{ and } \varphi \text{ is a germ of a local isomorphism as above}\}$ and the structure maps defined in the obvious way, we obtain a groupoid \mathcal{G} which is an orbifold structure on Y .

We say that the variety Y has *transversal ADE singularities* if the singular locus S is connected, smooth, and the pair (S, Y) is locally (in the complex topology) isomorphic to $(\mathbb{C}^k \times \{0\}, \mathbb{C}^k \times R)$. We have the following

Proposition 2.9. *Let Y be a variety with transversal ADE singularities. Then there is a unique complex holomorphic orbifold structure $[Y]$ on Y such that the fixed point locus of the local groups has codimension greater than 2.*

Proof. This is a particular case of the well known fact that every complex variety with quotient singularities has a unique orbifold structure such that the fixed point locus of the local groups has codimension greater than 2 (see e.g. [43]). \square

Definition 2.10. *An orbifold with transversal ADE singularities is the orbifold $[Y]$ associated to a variety Y with transversal ADE singularities as in Prop. 2.9.*

Notation 2.11. Let $[Y]$ be an orbifold with transversal ADE singularities. In the rest of the paper, we will use the presentation (\mathcal{G}, f) of $[Y]$ defined as follows. Let $y \in Y$ be a point. If $y \notin S$, take V_α to be a smooth open neighborhood of y , $U_\alpha := V_\alpha$ and $\chi_\alpha := \text{id}_{V_\alpha}$. If $y \in S$, then set V_α an open neighborhood of the form

$$\begin{aligned} V_\alpha &\cong \mathbb{C}^k \times R, \\ U_\alpha &:= \mathbb{C}^k \times \mathbb{C}^2, \\ G_\alpha &:= G \end{aligned}$$

and

$$\chi_\alpha : U_\alpha / G_\alpha \xrightarrow{\cong} V_\alpha,$$

where G_α acts on $U_\alpha := \mathbb{C}^k \times \mathbb{C}^2$ only on the second factor. The presentation of $[Y]$, (\mathcal{G}, f) , is constructed as explained in the beginning of the Section. The triple $(U_\alpha, G_\alpha, \chi_\alpha)$ is called *orbifold chart* at y .

Remark 2.12. If Y is a 3-fold with canonical singularities, then with the exception of at most a finite number of points, every point in Y has an open neighborhood which is nonsingular or isomorphic to $\mathbb{C} \times R$ [38].

3 Chen-Ruan cohomology

In this Section we compute the Chen-Ruan cohomology of orbifolds with transversal A_n singularities. As a vector space, the Chen-Ruan cohomology of $[Y]$ is defined by

$$H_{\text{CR}}^*([Y]) := \oplus_{(g) \in T} H^{*-2\iota(g)}(Y_{(g)}),$$

where $Y_{(g)}$ is the coarse moduli space of the twisted (untwisted) sector $[Y_{(g)}]$ ($[Y_{(1)}]$), T is the set of connected components of the inertia orbifold $[Y_1]$, and $\iota_{(g)}$ is the age (also called degree shifting) [11]. We work with cohomology with complex coefficients, so $H^*(Y_{(g)})$ denotes singular cohomology with complex coefficients.

The orbifold cup product \cup_{CR} is defined in terms of an obstruction bundle $[E]$, which is an orbifold vector bundle over the orbifold $[Y_3^0]$, the sub-orbifold of the orbifold of 3-multisectors corresponding to elements $(g_1, g_2, g_3) \in \mathcal{S}_{\mathcal{G}}^3$ such that $g_1 \cdot g_2 \cdot g_3 = 1$, [10] [11].

There is an orbifold morphism

$$[\tau] : [Y_1] \rightarrow [Y]$$

whose underlying continuous map is

$$\begin{aligned} \tau : Y_1 &\rightarrow Y \\ (y, (g)_y) &\mapsto y. \end{aligned}$$

3.1 Inertia orbifold and monodromy

We study some properties of the inertia orbifold of an orbifold $[Y]$ with transversal ADE singularities. The presentation of $[Y]$ described in Not. 2.11 will be used.

Lemma 3.1. *The orbifold $[Y]$ induces a natural orbifold structure on S .*

Proof. Let s and t be the source and target maps of \mathcal{G} , and denote by $F : G_0 \rightarrow Y$ the composition of the quotient map $G_0 \rightarrow |\mathcal{G}|$ followed by f . We define

$$H_0 := F^{-1}(S) \quad \text{and} \quad H_1 := t^{-1}(H_0).$$

Since $t^{-1}(H_0) = s^{-1}(H_0)$, we obtain a groupoid \mathcal{H} whose structure maps are the restriction of the structure maps of \mathcal{G} to H_0 and H_1 . The orbit space $|\mathcal{H}|$ is contained in $|\mathcal{G}|$ and the restriction of f to $|\mathcal{H}|$, $f|$, is an homeomorphism from $|\mathcal{H}|$ to S . Then $(\mathcal{H}, f|)$ is the orbifold structure on S . \square

Notation 3.2. We denote by $[S]$ the orbifold given by the equivalence class of $(\mathcal{H}, f|)$. $[S]$ can be viewed as sub-orbifold of $[Y]$. The normal vector bundle of $[S]$ in $[Y]$ is denoted by $[N]$.

Proposition 3.3. 1. *The restriction of $\tau : Y_1 \rightarrow Y$ to the coarse moduli space of the union of the twisted sectors, $\sqcup_{(g) \neq (1)} Y_{(g)}$, is a topological covering*

$$\tau| : \sqcup_{(g) \neq (1)} Y_{(g)} \rightarrow S.$$

2. *For any point $y \in S$, the fiber $(\tau|)^{-1}(y)$ is canonically identified with the set of conjugacy classes of the local group $G_y := (s, t)^{-1}(y, y)$ which are different from the class of the neutral element (1) , and hence with the set of the non trivial irreducible representations of G_y .*
3. *For $y \in S$, the fiber $[N]_y$ of the normal bundle of $[S]$ in $[Y]$ is a 2-dimensional representation of G_y , let Γ_{G_y} be the McKay graph of G_y with respect to $[N]_y$. Then, the monodromy of the covering $\tau|$ at y takes values in the automorphism group of the McKay graph Γ_{G_y} .*

Proof. 1. Following [10], we consider the following Cartesian diagram which defines $\mathcal{S}_{\mathcal{G}}$ and π

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{G}} & \longrightarrow & G_1 \\ \pi \downarrow & & \downarrow (s, t) \\ G_0 & \xrightarrow{\Delta} & G_0 \times G_0 \end{array} \quad (10)$$

where Δ is the diagonal. $\mathcal{S}_{\mathcal{G}}$ is a \mathcal{G} -space with action given by

$$\begin{aligned} G_1 \times_{s \times \pi} \mathcal{S}_{\mathcal{G}} &\rightarrow \mathcal{S}_{\mathcal{G}} \\ (a, b) &\mapsto aba^{-1} \end{aligned} \quad (11)$$

and the action-groupoid $\mathcal{G} \ltimes \mathcal{S}_{\mathcal{G}}$ is a presentation of the inertia orbifold $[Y_1]$.

Let $\pi|_{H_0} : \mathcal{S}_{\mathcal{G}|H_0} \rightarrow H_0$ be the base change of π with respect to the inclusion $H_0 \rightarrow G_0$ (H_0 is defined in the proof of the previous Lemma). With respect to our presentation of $[Y]$ (see Not. 2.11), we have

$$\mathcal{S}_{\mathcal{G}|H_0} \cong H_0 \times G.$$

The action of \mathcal{G} on $\mathcal{S}_{\mathcal{G}}$ restricts to an action on $H_0 \times (G - \{1\})$, which under the previous identification is described as follows

$$((u, \varphi, u'), (u, g)) \mapsto (u' = \varphi(u), \varphi \circ g \circ \varphi^{-1}). \quad (12)$$

The associated action-groupoid, $\mathcal{G} \ltimes (H_0 \times (G - \{1\}))$, is a presentation of $\sqcup_{(g) \neq (1)} [Y_{(g)}]$. The restriction of $\mathcal{G} \ltimes (H_0 \times (G - \{1\}))$ to $(U_{\alpha})^{G_{\alpha}} \times (G - \{1\})$ is isomorphic to the action groupoid

$$G \times ((U_{\alpha})^G \times (G - \{1\})) \rightrightarrows (U_{\alpha})^G \times (G - \{1\}),$$

moreover the orbifolds $[G \times ((U_{\alpha})^G \times (G - \{1\})) \rightrightarrows (U_{\alpha})^G \times (G - \{1\})]$ form an open covering of $\sqcup_{(g) \neq (1)} [Y_{(g)}]$. Thus we see that $(\tau|)^{-1}((U_{\alpha})^G)$ is disjoint union of copies of $(U_{\alpha})^G$ and the restriction of $\tau|$ on any of these components is an homeomorphism. This proves the statement.

2. It follows from diagram (10) and the action (11) that

$$(\tau|)^{-1}(y) = (\pi^{-1}(y) - \{\text{id}_y\}) / \pi^{-1}(y) = (G_y - \{\text{id}_y\}) / G_y$$

where G_y acts by conjugation. This establish the correspondence between fibers of $\tau|$ and conjugacy classes of local groups.

3. Let $y \in S$. Using the chart $(U_\alpha, G_\alpha, \chi_\alpha)$ at y we get an identification of the local group $G_{y'}$ with $G_\alpha = G$, for any $y' \in (U_\alpha)^G$. It is clear that these identifications respect the McKay graphs. It remains to show that, if $y \in V_\alpha \cap V_\beta$, the isomorphism $G_\alpha \cong G_\beta$ induced by the orbifold structure respects the McKay graphs.

We recall that, in this situation, if $W \subset U_\alpha$ and $W' \subset U_\beta$ are neighborhoods of $\chi_\alpha^{-1}(y)$ and $\chi_\beta^{-1}(y)$ respectively, and $\varphi : W \rightarrow W'$ is an isomorphism such that $\chi_\beta \circ \varphi = \chi_\alpha$, then there exists a unique isomorphism $\lambda : G_\alpha \rightarrow G_\beta$ such that φ is λ -equivariant [33]. We identify the representations of G_α with that of G_β by means of λ . In this way the irreducible representations correspond to irreducible representations. Finally, the linear map

$$T_{\chi_\alpha^{-1}(y)}\varphi : T_{\chi_\beta^{-1}(y)}U_\alpha \rightarrow T_{\chi_\beta^{-1}(y)}U_\beta$$

gives an isomorphism between the representations $N_{U_\alpha^G/U_\alpha}$ of G_α and $N_{U_\beta^G/U_\beta}$ of G_β . Now the statement follows from the definition of the McKay graph and of the monodromy of a topological cover, see e.g. [31]. \square

Definition 3.4. Let $[Y]$ be an orbifold with transversal ADE singularities, $y \in S$. The **monodromy** of $[Y]$ in y is the monodromy, in y , of the topological cover

$$\tau_1 : \sqcup_{(g) \neq (1)} Y_{(g)} \rightarrow S,$$

it is denoted by the group homomorphism

$$\mathfrak{m}_y : \pi_1(S, y) \rightarrow \text{Aut}(\tau_1^{-1}(y)).$$

Remark 3.5. For $G = A_n$, $n \geq 1$, D_n $n \geq 4$, E_6, E_7, E_8 (see Th. 2.2), the automorphism group of Γ_G is given as follows:

G	$\text{Aut}(\Gamma_G)$
A_1	$\{1\}$
$A_n \quad n \geq 2$	\mathbb{Z}_2
D_4	\mathfrak{S}_3
$D_n \quad n \geq 5$	\mathbb{Z}_2
E_6	\mathbb{Z}_2
E_7	$\{1\}$
E_8	$\{1\}$

where we have written on the left side the group G and on the right $\text{Aut}(\Gamma_G)$.

The previous considerations give constraints on the topology of the spaces $Y_{(g)}$ for $(g) \in T$. The following Corollary is an easy consequence of Prop. 3.3.

Corollary 3.6. Let $[Y]$ be an orbifold with transversal ADE singularities. Then, if the monodromy is trivial, all the coarse moduli spaces of the twisted sectors are canonically isomorphic to S .

If the monodromy is not trivial, there exists an open neighborhood U of S and a covering space $\tilde{U} \rightarrow U$ such that \tilde{U} has a structure of orbifold with transversal ADE singularities and trivial monodromy.

Proof. For any $(g) \neq (1)$, the map

$$\tau_{|Y_{(g)}} : Y_{(g)} \rightarrow S$$

is a connected topological covering. If $[Y]$ has trivial monodromy, then $\tau_{|Y_{(g)}}$ has also trivial monodromy. It follows that $\tau_{|Y_{(g)}}$ is an homeomorphism.

Assume now that the monodromy is not trivial. Let $U \subset Y$ be a tubular neighborhood of S and $y \in Y$ a point. Then the representation

$$\mathfrak{m}_y : \pi_1(S, y) \rightarrow \text{Aut}(\tau_1^{-1}(y))$$

guarantee the existence of a covering $\tilde{U} \rightarrow U$ with the same monodromy \mathfrak{m}_y . Since $\tilde{U} \rightarrow U$ is a local homeomorphism, \tilde{U} is a complex analytic space with transversal ADE singularities, hence it has a structure of orbifold with transversal ADE singularities $[\tilde{U}]$. By construction $[\tilde{U}]$ has trivial monodromy. \square

Remark 3.7. Notice that the twisted sectors $[Y_{(g)}]$ of $[Y]$ depend only on a neighborhood of S in Y . Indeed, let $U \subset Y$ be an open neighborhood of S in Y , then U is a variety with transversal ADE singularities and the twisted sectors $[U_{(g)}]$ of $[U]$ are canonically isomorphic to $[Y_{(g)}]$. So,

$$[Y_1] \cong [Y] \bigsqcup_{(g) \in T, (g) \neq (1)} [U_{(g)}].$$

Corollary 3.8. *Let $[Y]$ be an orbifold with transversal A_n singularities and trivial monodromy. If $n \geq 2$, then the normal bundle $[N]$ of $[S]$ in $[Y]$ is isomorphic to the direct sum of two line bundles $[N]^g$ and $[N]^{g^{-1}}$ on $[S]$,*

$$[N] \cong [N]^g \oplus [N]^{g^{-1}}.$$

Proof. A presentation of $[N]$ is given by the \mathcal{H} -space $N_{H_0/G_0} \rightarrow H_0$, [10]. The subset $\mathcal{S}_{\mathcal{G}|H_0}$ of G_1 (see (10)) acts on $N_{H_0/G_0} \rightarrow H_0$ fixing the source points. Because of our special presentation (\mathcal{G}, f) we have the identification

$$\mathcal{S}_{\mathcal{G}|H_0} \cong H_0 \times G \cong H_0 \times \mathbb{Z}_{n+1},$$

then

$$N_{H_0/G_0} \cong (N_{H_0/G_0})^g \oplus (N_{H_0/G_0})^{g^{-1}},$$

where $g : \mathbb{Z}_{n+1} \rightarrow \mathbb{C}^*$ is a generator of the group of characters of \mathbb{Z}_{n+1} , and \mathbb{Z}_{n+1} acts on each factor by multiplication with the corresponding character.

In general, $(N_{H_0/G_0})^g \rightarrow H_0$ and $(N_{H_0/G_0})^{g^{-1}} \rightarrow H_0$ are not \mathcal{H} -spaces. However, if the monodromy is trivial, we identify the local groups G_y with \mathbb{Z}_{n+1} in such a way that, for any $(u, g) \in H_0 \times \mathbb{Z}_{n+1}$ and $(u, \varphi, u') \in G_1$,

$$\varphi \circ g \circ \varphi^{-1} = g.$$

Now, let $s, t : H_1 \rightarrow H_0$ be source and target maps of \mathcal{H} . The previous considerations imply that the map

$$\begin{aligned} \Phi : s^*(N_{H_0/G_0})^g &\rightarrow t^*(N_{H_0/G_0})^g \\ ((u, \varphi, u'), v) &\mapsto T\varphi(v) \end{aligned}$$

is an isomorphism of vector bundles over H_1 . Φ is compatible with the multiplication of the groupoid, hence $(N_{H_0/G_0})^g$ defines the orbifold line bundle $[N]^g$. In the same way, $(N_{H_0/G_0})^{g^{-1}}$ defines $[N]^{g^{-1}}$. \square

3.2 Chen-Ruan cohomology ring

We now describe the Chen-Ruan cohomology ring of an orbifold $[Y]$ with transversal A_n singularities. We first study the case $n = 1$. In this case, there is only one twisted sector which is isomorphic to $[S]$. Then, as a vector space, the Chen-Ruan cohomology is given by

$$H_{\text{CR}}^*([Y]) = H^*(Y) \oplus H^{*-2}(S)\langle e \rangle.$$

The obstruction bundle has rank zero (see e.g. [18]), so its top Chern class is 1. Then

$$(\delta_1 + \alpha_1 e) \cup_{\text{CR}} (\delta_2 + \alpha_2 e) = \delta_1 \cup \delta_2 + \frac{1}{2} i_* (\alpha_1 \cup \alpha_2) + (i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2))e$$

where $\delta_1 + \alpha_1 e, \delta_2 + \alpha_2 e \in H^*(Y) \oplus H^{*-2}(S)\langle e \rangle$. This can be deduced e.g. from the Decomposition Lemma 4.1.4 in [11].

Case A_n with $n \geq 2$ and trivial monodromy.

We will use the following convention.

Convention 3.9. Since the monodromy is trivial, we identify the local groups G_y with \mathbb{Z}_{n+1} . We use both the additive and multiplicative notations for the group operation.

Notation 3.10. The orbifold cup product can be described in terms of the Chern classes of $[N]^g$ and $[N]^{g^{-1}}$. But for later use we find more convenient to describe it in a different way. Consider the morphism

$$f : [S] \rightarrow S$$

that, naively speaking, forgets the orbifold structure. It is easy to see that

$$([N]^g)^{\otimes n+1} \cong f^* M, \quad ([N]^{g^{-1}})^{\otimes n+1} \cong f^* L \quad \text{and} \quad [N]^g \otimes [N]^{g^{-1}} \cong f^* K, \quad (13)$$

for some line bundles M, L and K on S . The orbifold cup product will be expressed in terms of the Chern classes of M, L and K .

Notation 3.11. From Cor. 3.6 we have that the topological space $Y_{(a)}$ underlying the non-twisted sector $[Y_{(a)}]$ is canonically homeomorphic to S , where $a \in \{1, \dots, n+1\}$. In particular the cohomology group $H^*(Y_{(a)})$ is identified with $H^*(S)$. We will denote $H^*(Y_{(a)})$ by $H^*(S)\langle e_a \rangle$.

Theorem 3.12. *Let $[Y]$ be an orbifold with transversal A_n singularities. Assume that the monodromy is trivial. Then, as a vector space*

$$H_{\text{CR}}^*([Y]) \cong H^*(Y) \oplus_{a=1}^{n+1} H^{*-2}(S)\langle e_a \rangle. \quad (14)$$

The orbifold cup product is skewsymmetric and it is given as follows:

1. $\alpha \cup_{\text{CR}} \beta = \alpha \cup \beta \in H^*(Y)$ if $\alpha, \beta \in H^*(Y)$
2. $e_a \cup_{\text{CR}} \beta = i^*(\beta)e_a \in H^*(S)$ if $\beta \in H^*(Y)$
3. $e_a \cup_{\text{CR}} e_b = \frac{1}{n+1}i_*([S]) \in H^*(Y)$ if $a + b = 0 \pmod{n+1}$
4. $e_a \cup_{\text{CR}} e_b = \frac{1}{n+1}c_1(L)e_{a+b}$ if $a + b < n + 1$
5. $e_a \cup_{\text{CR}} e_b = \frac{1}{n+1}c_1(M)e_{a+b-n-1}$ if $a + b > n + 1$,

where L and M are the line bundles defined by equations (13), $i : S \rightarrow Y$ is the inclusion of the singular locus in Y and $[S] \in H^0(S)$.

Proof. Equation (14) is a direct consequence of Prop. 3.6. The skewsymmetry of \cup_{CR} follows from the fact that $[Y]$ is Gorenstein. Finally, the description of \cup_{CR} follows from the Decomposition Lemma 4.1.4. in [11] and a formula for the obstruction bundle $[Y]$ that we explain now.

Following [10] we set

$$\mathcal{S}_0^3 := \{(a_1, a_2, a_3) \in G_1^3 | s(a_1) = t(a_1) = s(a_2) = t(a_2) = s(a_3) = t(a_3), a_1 \cdot a_2 \cdot a_3 = 1\}.$$

The anchor map is defined as

$$\begin{aligned} \pi_3 : \mathcal{S}_0^3 &\rightarrow G_0 \\ (a_1, a_2, a_3) &\mapsto s(a_1), \end{aligned}$$

and the groupoid \mathcal{G} acts on \mathcal{S}_0^3 as follows

$$\begin{aligned} G_1 \times_{s \times \pi_3} \mathcal{S}_0^3 &\rightarrow \mathcal{S}_0^3 \\ (b, (a_1, a_2, a_3)) &\mapsto (b \cdot a_1 \cdot b^{-1}, b \cdot a_2 \cdot b^{-1}, b \cdot a_3 \cdot b^{-1}). \end{aligned}$$

The action groupoid $\mathcal{G} \ltimes \mathcal{S}_0^3$ is a presentation for the orbifold $[Y_0^3]$. We have a decomposition of $[Y_0^3]$ as disjoint union of its connected components:

$$[Y_0^3] = \sqcup_{(\underline{a}) \in T_0^3} [Y_{(\underline{a})}],$$

where $\underline{a} := (a_1, a_2, a_3)$ and T_0^3 is the set of connected components of $[Y_0^3]$. Let $\mathcal{S}_{(\underline{a})}$ be the pre-image of $[Y_{(\underline{a})}]$ with respect to the natural map $\mathcal{S}_0^3 \rightarrow [Y_0^3]$. Then the action of \mathcal{G} on \mathcal{S}_0^3 restricts to an action on $\mathcal{S}_{(\underline{a})}$ giving a presentation for $[Y_{(\underline{a})}]$. We denote by $[E_{(\underline{a})}]$ the restriction of the obstruction bundle $[E]$ to $[Y_{(\underline{a})}]$.

If $a_1 = 0, a_2 = 0$ or $a_3 = 0$, then $[E_{(\underline{a})}]$ has rank 0, [11] Lemma 4.2.2, [18] Lemma 1.12. Hence, it remains to consider the case where $(a_1, a_2, a_3) \neq (0, 0, 0)$. Under the hypothesis of trivial monodromy and with our choice of \mathcal{G} , it follows that

$$T_0^3 = \{(a_1, a_2, a_3) \in \mathbb{Z}_{n+1}^3 | a_1 + a_2 + a_3 = 0\},$$

and

$$\mathcal{S}_{(\underline{a})} \cong H_0 \times \{(\underline{a})\}.$$

Let $\Sigma \rightarrow \mathbb{P}^1$ be the Galois cover of \mathbb{P}^1 , with Galois group the subgroup $\langle (\underline{a}) \rangle$ of \mathbb{Z}_{n+1} generated by a_1, a_2, a_3 , branched over $0, 1, \infty \in \mathbb{P}^1$, and with monodromy a_1, a_2, a_3 at $0, 1, \infty$ respectively. Then, $[E_{(\underline{a})}]$ has the following presentation

$$\left(H^1(\Sigma, \mathcal{O}_\Sigma) \otimes (\pi_3^* TG_0)|_{\mathcal{S}_{(\underline{a})}} \right)^{\langle (\underline{a}) \rangle} \rightarrow \mathcal{S}_{(\underline{a})},$$

where $()^{\langle (\underline{a}) \rangle}$ means the $\langle (\underline{a}) \rangle$ -invariant part with respect to the action on both factors. We replace now, in the previous expression, TG_0 with the normal bundle N_{H_0/G_0} , and Σ with the Galois cover

$C \rightarrow \mathbb{P}^1$ with Galois group \mathbb{Z}_{n+1} induced by the inclusion $\langle(\underline{a})\rangle \subset \mathbb{Z}_{n+1}$. We get the following presentation for $[E_{(\underline{a})}]$:

$$(H^1(C, \mathcal{O}_C) \otimes (\pi_3^* N_{H_0/G_0}))^{\mathbb{Z}_{n+1}} \rightarrow \mathcal{S}_{(\underline{a})}. \quad (15)$$

Notice that $p : C \rightarrow \mathbb{P}^1$ is an abelian cover in the sense of [35], so

$$p_* \mathcal{O}_C = \bigoplus_{\mathfrak{c} \in \mathbb{Z}_{n+1}^*} (L^{-1})^{\mathfrak{c}}$$

where \mathbb{Z}_{n+1}^* is the group of characters of \mathbb{Z}_{n+1} and \mathbb{Z}_{n+1} acts on $(L^{-1})^{\mathfrak{c}}$ via the character \mathfrak{c} . Hence, (15) becomes

$$\left(H^1(C, \mathcal{O}_C) \otimes (\pi_3^* N_{H_0/G_0})|_{\mathcal{S}_{(\underline{a})}} \right)^{\mathbb{Z}_{n+1}} \quad (16)$$

$$\cong \left(H^1(\mathbb{P}^1, (L^{-1})^{\mathfrak{g}}) \otimes (N_{H_0/G_0})^{\mathfrak{g}^{-1}} \right) \oplus \left(H^1(\mathbb{P}^1, (L^{-1})^{\mathfrak{g}^{-1}}) \otimes (N_{H_0/G_0})^{\mathfrak{g}} \right). \quad (17)$$

By Prop. 2.1 of [35], see also [10], we have that

$$L^{\mathfrak{g}} = \begin{cases} \mathcal{O}(2) & \text{if } a_1 + a_2 < n+1, \\ \mathcal{O}(1) & \text{if } a_1 + a_2 \geq n+1 \end{cases}$$

and

$$L^{\mathfrak{g}^{-1}} = \begin{cases} \mathcal{O}(1) & \text{if } a_1 + a_2 \leq n+1, \\ \mathcal{O}(2) & \text{if } a_1 + a_2 > n+1 \end{cases}$$

This concludes the proof. \square

The general case.

We now study the case in which the monodromy is not trivial. We first notice that it is enough to compute the Chen-Ruan cohomology ring

$$H_{\text{CR}}^*([U]),$$

where $U \subset Y$ is any open connected neighborhood of S . By Cor. 3.6 there exists a U and a \mathbb{Z}_2 -covering

$$p : \tilde{U} \rightarrow U$$

such that $[\tilde{U}]$ has trivial monodromy. There is a unique morphism of orbifolds

$$[p] : [\tilde{U}] \rightarrow [U]$$

with associated continuous map p . Moreover we have a morphism between the inertia orbifolds:

$$[p_1] : [\tilde{U}_1] \rightarrow [U_1].$$

The group \mathbb{Z}_2 acts on $H_{\text{CR}}^*([\tilde{U}])$ and the morphism

$$p_1^* : H_{\text{CR}}^*([U]) \rightarrow H_{\text{CR}}^*([\tilde{U}])$$

induces an isomorphism between $H_{\text{CR}}^*([U])$ and $\left(H_{\text{CR}}^*([\tilde{U}]) \right)^{\mathbb{Z}_2}$, as vector spaces. We will denote by p_1^* this isomorphism.

Proposition 3.13. *The restriction of the orbifold cup product to $\left(H_{\text{CR}}^*([\tilde{U}]) \right)^{\mathbb{Z}_2}$ defines an associative product such that*

$$p_1^* : H_{\text{CR}}^*([U]) \rightarrow \left(H_{\text{CR}}^*([\tilde{U}]) \right)^{\mathbb{Z}_2}$$

is a ring isomorphism.

Proof. Cor. 3.6 imply that we can identify the coarse moduli space of $[\tilde{U}_1]$ as follows

$$\tilde{U}_1 \cong \tilde{U} \sqcup_{a \in \mathbb{Z}_{n+1} - \{0\}} \tilde{S} \times \{a\},$$

where $\tilde{S} := p^{-1}(S)$. \mathbb{Z}_2 acts on \tilde{U}_1 by the monodromy of $p : \tilde{U} \rightarrow U$ on \tilde{U} , and on $\sqcup_{a \in \mathbb{Z}_{n+1} - \{0\}} \tilde{S} \times \{a\}$ through

$$\begin{aligned} \epsilon : \sqcup_{a \in \mathbb{Z}_{n+1} - \{0\}} \tilde{S} \times \{a\} &\rightarrow \sqcup_{a \in \mathbb{Z}_{n+1} - \{0\}} \tilde{S} \times \{a\} \\ (\tilde{y}, a) &\mapsto (\epsilon \cdot \tilde{y}, -a) \end{aligned}$$

where $\tilde{y} \mapsto \epsilon \cdot \tilde{y}$ is the monodromy on \tilde{S} . This action induces an action of \mathbb{Z}_2 on \tilde{U}_1^0 in a natural way. From the description of the obstruction bundle $[E] \rightarrow [\tilde{U}_1^0]$ as given in the proof of Th. 3.12 it follows that

$$\epsilon^*[E] \cong [E]$$

for $\epsilon \in \mathbb{Z}_2$. Then the result follows. \square

3.3 Examples

We give here some special examples of Chen-Ruan cohomology rings.

Example 3.14. : surface case. Let Y be a projective surface with one singular point of type A_n . So $S = \{p\}$ is a point and a neighborhood U of $p \in Y$ is isomorphic to

$$U \cong \{(x, y, z) \in \mathbb{C}^3 : xy - z^{n+1} = 0\}.$$

Here U is the quotient of \mathbb{C}^2 by the action of the group μ_{n+1} given by $\epsilon \cdot (u, v) = (\epsilon \cdot u, \epsilon^{-1} \cdot v)$, $\epsilon \in \mu_{n+1}$.

As a vector space

$$H_{\text{CR}}^*([Y]) = H^*(Y) \oplus H^{*-2}(S)\langle e_1 \rangle \oplus \dots \oplus H^{*-2}(S)\langle e_n \rangle.$$

The product rule is given by

$$e_i \cup_{\text{CR}} e_j = \begin{cases} 0 & \text{if } i+j \neq 0 \pmod{n+1}, \\ \frac{1}{n+1} i_*[S] \in H^4(Y) & \text{if } i+j = 0 \pmod{n+1}. \end{cases}$$

Example 3.15. : transversal A_2 -case, trivial monodromy. In this case we have

$$H_{\text{CR}}^*([Y]) = H^*(Y) \oplus H^{*-2}(S)\langle e_1 \rangle \oplus H^{*-2}(S)\langle e_2 \rangle$$

as a vector space. Given $\delta_1 + \alpha_1 e_1 + \beta_1 e_2, \delta_2 + \alpha_2 e_1 + \beta_2 e_2 \in H_{\text{CR}}^*(Y)$, the following expression for the orbifold cup product holds:

$$\begin{aligned} (\delta_1 + \alpha_1 e_1 + \beta_1 e_2) \cup_{\text{CR}} (\delta_2 + \alpha_2 e_1 + \beta_2 e_2) &= \delta_1 \cup \delta_2 + \frac{1}{2} i_*(\alpha_1 \cup \beta_2 + \beta_1 \cup \alpha_2) + \\ &\quad (i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2) + \beta_1 \cup \beta_2 \cup c_1(L))e_1 + \\ &\quad (i^*(\delta_1) \cup \beta_2 + \beta_1 \cup i^*(\delta_2) + \alpha_1 \cup \alpha_2 \cup c_1(M))e_2. \end{aligned}$$

4 Crepant resolution

In this Section we show that any variety with transversal ADE singularities Y has a unique crepant resolution $\rho : Z \rightarrow Y$. Then we restrict our attention to the A_n -case and trivial monodromy. In this case we describe the exceptional locus E in terms of the line bundles L, M and K defined in Not. 3.10. Finally we compute the cohomology ring $H^*(Z)$ of Z in terms of the cohomology of Y and of the Chern classes of L, M and K .

4.1 Existence and unicity

First observe that if $R \subset \mathbb{C}^3$ is a rational double point. Then R has a unique crepant resolution $\rho : \tilde{R} \rightarrow R$, where \tilde{R} can be obtained by blowing-up successively the singular locus. The exceptional locus $C \subset \tilde{R}$ is the union of rational curves C_l with self-intersection $C_l \cdot C_l = -2$. The shape of C inside \tilde{R} is described by the resolution graph (Section 2.1).

Example 4.1. (*Resolution of A_n -surface singularities*). Let

$$R = \{(x, y, z) \in \mathbb{C}^3 : xy - z^{n+1} = 0\}$$

be a surface singularity of type A_n . Let $r : R_1 = Bl_0 R \rightarrow R$ be the blow-up of R at the origin. Then R_1 is covered by three open affine varieties U, V and W , where

$$\begin{aligned} U &= \left\{ \left(x, \frac{v}{u}, \frac{w}{u} \right) \in \mathbb{C}^3 : \left(\frac{v}{u} \right) - x^{n-1} \left(\frac{w}{u} \right)^{n+1} = 0 \right\} \\ V &= \left\{ \left(y, \frac{u}{v}, \frac{w}{v} \right) \in \mathbb{C}^3 : \left(\frac{u}{v} \right) - y^{n-1} \left(\frac{w}{v} \right)^{n+1} = 0 \right\} \\ W &= \left\{ \left(z, \frac{u}{w}, \frac{v}{w} \right) \in \mathbb{C}^3 : \frac{u}{w} \frac{v}{w} - z^{n-1} = 0 \right\}. \end{aligned}$$

and the restriction of r to U, V, W is given by

$$\begin{aligned} r|_U : \left(x, \frac{v}{u}, \frac{w}{u} \right) &\mapsto \left(x, x \frac{v}{u}, x \frac{w}{u} \right) = (x, y, z) \\ r|_V : \left(y, \frac{u}{v}, \frac{w}{v} \right) &\mapsto \left(y \frac{u}{v}, y, y \frac{w}{v} \right) = (x, y, z) \\ r|_W : \left(z, \frac{u}{w}, \frac{v}{w} \right) &\mapsto \left(z \frac{u}{w}, z \frac{v}{w}, z \right) = (x, y, z). \end{aligned}$$

If $n = 1$, R_1 is smooth and the exceptional locus is given by one rational curve C . A direct computation shows that $C \cdot C = -2$. If $n \geq 2$, R_1 has a singularity of type A_{n-2} at the origin of W and the exceptional locus is the union of two rational curves meeting at the singular point. Then, after a finite number of blow-ups, we get a smooth surface.

Let \tilde{R} be the first smooth surface obtained in this way, $\rho : \tilde{R} \rightarrow R$ the composition of the blow-up morphisms and $C = C_1, \dots, C_n$ be the components of the exceptional locus. Then $C_l \cdot C_l = -2$ for any $l \in \{1, \dots, n\}$ and moreover there exists an isomorphism of sheaves

$$\rho^* K_R \cong K_{\tilde{R}}.$$

Hence $\rho : \tilde{R} \rightarrow R$ is crepant.

We have the following result.

Proposition 4.2. *Let Y be a variety with transversal ADE singularities. Then Y has a unique crepant resolution $\rho : Z \rightarrow Y$ up to isomorphism.*

Proof. To prove the existence, one can proceed as follows. Let $r : \text{Bl}_S Y \rightarrow Y$ be the blow-up of Y along S . If $\text{Bl}_S Y$ is smooth, then define $Z := \text{Bl}_S Y$ and $\rho = r$. Otherwise, blow-up again. As in the surface case, after a finite number of blow-up, we will find a smooth variety. Define Z to be the first smooth variety obtained in this way, and ρ be the composition of the blow-up morphisms. We now show that $\rho^* K_Y \cong K_Z$. In general we have

$$\rho^* K_Y \cong K_Z + \sum_{l=1}^n a_l E_l,$$

where E_l are the components of the exceptional divisor E of ρ and a_l are integers defined as follows. Let $z \in E_l$ be a generic point, and $g_l = 0$ be an equation for E_l in a neighborhood of z . Let s be a (local) generator of K_Y in a neighborhood of $\rho(z)$. Then a_l is defined by the equation

$$\rho^*(s) = g_l^{a_l} \cdot (dz_1 \wedge \dots \wedge dz_d),$$

where z_1, \dots, z_d are local coordinates for Z in z [12]. In our case, Y is locally a product $R \times \mathbb{C}^k$, so Z is locally isomorphic to $\tilde{R} \times \mathbb{C}^k$, with $k = d - 2$. Then, since $\tilde{R} \rightarrow R$ is crepant, $a_l = 0$ for all $l \in \{1, \dots, d\}$.

We now prove unicity. Assume that $\rho_1 : Z_1 \rightarrow Y$ is another crepant resolution of Y . By [19], Lemma 2.10, the exceptional locus of ρ_1 is of pure codimension 1 in Z_1 . Let $I_{S/Y}$ be the ideal sheaf of S in Y . The sheaf $J := \rho_1^{-1}(I_{S/Y}) \cdot \mathcal{O}_{Z_1}$ is the ideal sheaf of the exceptional locus of ρ_1 , hence it is invertible. Moreover, we get a morphism $Z_1 \rightarrow \text{Bl}_S Y$ which lifts ρ_1 [23]. Repeating this argument we get a morphism $f : Z_1 \rightarrow Z$. To see that f is an isomorphism we notice that the morphism

$$\wedge^d T_{Z_1} \rightarrow f^* \wedge^d T_Z$$

is an isomorphism since it corresponds to a non zero global section of $\mathcal{O}_{Z_1}(K_{Z_1} - f^* K_Z) \cong \mathcal{O}_{Z_1}$. This shows that f is a local isomorphism, and since it is birational, it is one to one. \square

4.2 Geometry of the exceptional divisor

We now restrict our attention to varieties with transversal A_n singularities such that the associated orbifold $[Y]$ has trivial monodromy. In this case any component of the exceptional divisor has a structure of \mathbb{P}^1 -bundle on S and we describe it as the projectivization of a vector bundle of rank 2. These vector bundles will be defined in terms of the line bundles L , M and K previously introduced. This will allow us to give a description of the cohomology of Z in terms of the Chern classes of L , M and K so that we can compare the Chen-Ruan cohomology ring $H_{\text{CR}}^*([Y])$ (Th. 3.12) with the cohomology ring $H^*(Z)$.

Notation 4.3. From now on Y denotes a variety with transversal A_n singularities such that the associated orbifold $[Y]$ has trivial monodromy. The crepant resolution obtained by blowing-up the singular locus $i : S \rightarrow Y$ is denoted by $\rho : Z \rightarrow Y$. The exceptional divisor will be denoted by E and by $j : E \rightarrow Z$ the inclusion, the restriction of ρ to E by $\pi : E \rightarrow S$.

For every vector bundle F over the variety X , by $\mathbb{P}(F)$ we denote the projective bundle of lines in F as defined in [20] Appendix B.5.5 (and therein denoted by $\text{P}(F)$). On $\mathbb{P}(F)$ there is a canonical line bundle $\mathcal{O}_F(1)$. So, for any integer $m \in \mathbb{Z}$, we have the line bundle $\mathcal{O}_F(m)$ on $\mathbb{P}(F)$, for further details see [20].

The A_1 case

Proposition 4.4. *Let Y be a variety with transversal A_1 singularities. Then E is irreducible and there exist two vector bundles F and G on S with rank 2 and 1 respectively such that*

$$\begin{aligned} E &\cong \mathbb{P}(F), \\ N_{E/Z} &\cong \mathcal{O}_F(-2) \otimes \pi^*G. \end{aligned} \tag{18}$$

Moreover F and G are related by

$$\wedge^2 F \otimes G \cong R^1 \pi_* N_{E/Z}. \tag{19}$$

Proof. We have that $Z = Bl_S Y$ and the normal cone $C_S Y$ of S in Y is a conic bundle with fiber isomorphic to $\{(x, y, z) \in \mathbb{C}^3 : xy - z^2 = 0\}$. Therefore $\pi : E = \mathbb{P}(C_S Y) \rightarrow S$ is a \mathbb{P}^1 bundle over S and in particular it is irreducible. Since S is smooth, there exists a rank two vector bundle F on S such that $E \cong \mathbb{P}(F)$. Let us fix one of these bundles and denote it by F . The normal bundle $N_{E/Z}$ is a line bundle whose restriction on each fiber $\pi^{-1}(s)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$, then (18) follows. Using the projection formula (see e.g. [23]), we have (19). \square

The case $n \geq 2$

Notation 4.5. Since the monodromy is trivial, the exceptional divisor E of $\rho : Z \rightarrow Y$ has n irreducible components. We denote such components by E_1, \dots, E_n in such a way that

$$E_l \cap E_m = \begin{cases} \emptyset & \text{if } |l - m| > 1, \\ \neq \emptyset & \text{if } |l - m| = 1. \end{cases}$$

The restriction of $\pi : E \rightarrow S$ to E_l is denoted by $\pi_l : E_l \rightarrow S$, and the restriction of $j : E \rightarrow Z$ to E_l by $j_l : E_l \rightarrow Z$, for $l \in \{1, \dots, n\}$. We denote by β_l the generic fiber of π_l .

Proposition 4.6. *There are line bundles L_l, M_l on S , for $l \in \{1, \dots, n\}$, such that:*

- a.** *for any $l \in \{1, \dots, n\}$ there is an isomorphism $E_l \cong \mathbb{P}(L_l \oplus M_l)$ (which we fix for the rest of the paper);*
- b.** *$L_l \otimes M_l^\vee \cong M \otimes (K^\vee)^{\otimes l}$, for all $l \in \{1, \dots, n\}$;*
- c.** *under the identification of E_l with $\mathbb{P}(L_l \oplus M_l)$ in **a**, we have the following description of the intersection locus of two components of E :*

$$E_k \cap E_l = \begin{cases} \emptyset & \text{if } |k - l| > 1, \\ \mathbb{P}(M_{l-1}) \subset E_{l-1} & \text{if } k = l - 1, \\ \mathbb{P}(L_l) \subset E_l & \text{if } k = l + 1. \end{cases}$$

Proof. We prove the Proposition in the following way: we identify Z with the variety obtained from Y after a finite number of blow-ups; we will show that at each blow-up the normal cone to the singular locus is the union of two vector bundles of rank two over S ; finally we describe these vector bundles in terms of L, M and K .

Let (\mathcal{G}, f) be the presentation of $[Y]$ described in Not. 2.11. We identify Y with the orbit space $|\mathcal{G}|$ through f . For any component U_α of G_0 , we denote with $(\underline{w}_\alpha, u_\alpha, v_\alpha)$ the standard coordinate system for U_α . Then $\chi_\alpha : U_\alpha \rightarrow V_\alpha$ is given as follows

$$\chi_\alpha(\underline{w}_\alpha, u_\alpha, v_\alpha) = (\underline{w}_\alpha, u_\alpha^{n+1}, v_\alpha^{n+1}, u_\alpha \cdot v_\alpha) =: (\underline{w}_\alpha, x_\alpha, y_\alpha, z_\alpha).$$

For any $y \in S$, let $u \in U_\alpha$ and $u' \in U_\beta$ be points over y , i.e. $\chi_\alpha(u) = \chi_\beta(u') = y$. Let $\varphi_{\alpha\beta}$ be the \mathbb{Z}_{n+1} -equivariant isomorphisms between neighborhoods of u and u' such that $\varphi_{\alpha\beta}(u) = u'$, and $\Phi_{\alpha\beta}, F_{\alpha\beta}, G_{\alpha\beta}$ the components of $\varphi_{\alpha\beta}$ with respect to the coordinates $(\underline{w}_\beta, u_\beta, v_\beta)$. Since $\varphi_{\alpha\beta}$ is \mathbb{Z}_{n+1} -equivariant, we have the following change of variable expression

$$\begin{aligned} x_\beta &= x_\alpha \left(\frac{\partial F_{\alpha\beta}}{\partial u_\alpha} \right)^{n+1} + \text{higher order terms} \\ y_\beta &= y_\alpha \left(\frac{\partial G_{\alpha\beta}}{\partial v_\alpha} \right)^{n+1} + \text{h.o.t.} \\ z_\beta &= z_\alpha \frac{\partial F_{\alpha\beta}}{\partial u_\alpha} \frac{\partial G_{\alpha\beta}}{\partial v_\alpha} + \text{h.o.t.} \end{aligned}$$

Notice that $\left(\frac{\partial F_{\alpha\beta}}{\partial u_\alpha} \right)^{n+1}$, $\left(\frac{\partial G_{\alpha\beta}}{\partial v_\alpha} \right)^{n+1}$ and $\frac{\partial F_{\alpha\beta}}{\partial u_\alpha} \frac{\partial G_{\alpha\beta}}{\partial v_\alpha}$ are transition functions for M, L and K respectively (Not. 3.10). To conclude, we distinguish two cases: n even and n odd.

n even

From the previous considerations it is clear that the normal cone of S in Y is the union of two irreducible components, C_1 and C_2 . Moreover C_1 and C_2 have a structure of vector bundles of rank 2 over S and they are given by

$$\begin{aligned} C_1 &\cong M \oplus K \\ C_2 &\cong L \oplus K. \end{aligned}$$

Furthermore the intersection $C_1 \cap C_2$ in $C_S Y$ is given by the line bundle K . Then we define $L_1 := M$, $M_1 = L_n := K$, $M_n := L$.

If $n = 2$ the result holds, otherwise $Bl_S Y$ is a variety over Y with transversal A_{n-2} singularities, the exceptional divisor is $\mathbb{P}(C_S Y) = \mathbb{P}(C_1) \cup \mathbb{P}(C_2)$, the singular locus is $\mathbb{P}(C_1) \cap \mathbb{P}(C_2)$. Let $(a_\alpha, b_\alpha, z_\alpha)$, $(a_\beta, b_\beta, z_\beta)$ be coordinates in a neighborhood of the singular locus. The blow-up morphism, in these coordinates, is given by: $x_\alpha = a_\alpha z_\alpha$, $y_\alpha = b_\alpha z_\alpha$, $z_\alpha = z_\alpha$. The two systems of coordinates, $(a_\alpha, b_\alpha, z_\alpha)$ and $(a_\beta, b_\beta, z_\beta)$ are related as follows:

$$\begin{aligned} a_\beta &= \frac{x_\beta}{z_\beta} = \frac{F_{\alpha\beta}^{n+1}(a_\alpha z_\alpha)}{F_{\alpha\beta} \cdot G_{\alpha\beta}} \\ b_\beta &= \frac{y_\beta}{z_\beta} = \frac{G_{\alpha\beta}^{n+1}(a_\alpha z_\alpha)}{F_{\alpha\beta} \cdot G_{\alpha\beta}} \\ z_\beta &= F_{\alpha\beta} \cdot G_{\alpha\beta}. \end{aligned} \tag{20}$$

Notice that, on the right hand side of the first two equations, both numerator and denominator are multiples of z_α . So, after dividing by z_α , (20) becomes

$$\begin{aligned} a_\beta &= a_\alpha \frac{\left(\frac{\partial F_{\alpha\beta}}{\partial u_\alpha}\right)^{n+1}}{\left(\frac{\partial F_{\alpha\beta}}{\partial u_\alpha} \frac{\partial G_{\alpha\beta}}{\partial v_\alpha}\right)} + \text{h.o.t.'s} \\ b_\beta &= b_\alpha \frac{\left(\frac{\partial G_{\alpha\beta}}{\partial v_\alpha}\right)^{n+1}}{\left(\frac{\partial F_{\alpha\beta}}{\partial u_\alpha} \frac{\partial G_{\alpha\beta}}{\partial v_\alpha}\right)} + \text{h.o.t.'s} \\ z_\beta &= F_{\alpha\beta} \cdot G_{\alpha\beta}. \end{aligned}$$

Then the normal cone of the singular locus, after the first blow-up, is the union of the irreducible components $(M \otimes K^\vee) \oplus K$ and $K \oplus (L \otimes K^\vee)$ intersecting along K .

Under the identification of the strict transform of $\mathbb{P}(C_S Y)$ with $\mathbb{P}(M \oplus K) \cup \mathbb{P}(K \oplus L)$, we have that $\mathbb{P}((M \otimes K^\vee) \oplus K) \cap \mathbb{P}(M \oplus K) = \mathbb{P}(K) \subset \mathbb{P}(M \oplus K)$, $\mathbb{P}((M \otimes K^\vee) \oplus K) \cap \mathbb{P}(M \oplus K) = \mathbb{P}(M \otimes K^\vee) \subset \mathbb{P}(M \otimes K^\vee \oplus K)$ and $\mathbb{P}(M \oplus K) \cap \mathbb{P}(K \oplus (L \otimes K^\vee)) = \emptyset$. We set $L_2 := M \otimes K^\vee$, $M_2 = L_{n-1} := K$ and $M_{n-1} := L \otimes K^\vee$. Proceeding in this way, after $k = n/2$ steps we get the result.

$n = 2k + 1$ odd

We can identify E_{k+1} with $\mathbb{P}(L_{k+1} \oplus M_{k+1})$ in such a way that point \mathbf{c} of the proposition is verified. The only thing we have to show is that

$$L_{k+1} \otimes M_{k+1}^\vee \cong M \otimes (K^\vee)^{\otimes k+1}.$$

This can be seen in the following way. Write

$$N_{E_{k+1}/Z} = \mathcal{O}_{L_{k+1} \oplus M_{k+1}}(-2) \otimes \pi_{k+1}^* G$$

for some line bundle G on S , so

$$N_{E_{k+1}/Z|_{\mathbb{P}(L_{k+1})}} \cong L_{k+1}^{\otimes 2} \otimes G.$$

On the other hand,

$$N_{E_{k+1}/Z|_{\mathbb{P}(L_{k+1})}} \cong N_{E_k \cap E_{k+1}/E_k} \cong L_k \otimes M_k^\vee \cong M \otimes (K^{\otimes k})^\vee$$

(see e.g. [20], Appendix B.5.6). So we get the relation

$$L_{k+1}^{\otimes 2} \otimes G \cong M \otimes (K^{\otimes k})^\vee.$$

The same considerations for $k + 1$ give the relation

$$M_{k+1}^{\otimes 2} \otimes G \cong K^{\otimes k+2} \otimes M^\vee.$$

This prove the assertion. \square

4.3 Cohomology ring of the crepant resolution

Notation 4.7. In this Section we use Not. 4.5. Moreover, by abuse of notation, for any variety X and line bundle L on X , we will denote by L the first Chern class $c_1(L) \in H^2(X)$. If $\alpha \in H^*(X)$, then the cup product $\alpha \cup c_1(L) \in H^*(X)$ will be denoted by αL .

The A_1 -case

Proposition 4.8. *Let Y be a variety with transversal A_1 singularities. Then the following map is an isomorphism of vector spaces*

$$\begin{aligned} H^*(Y) \oplus H^{*-2}(S)\langle E \rangle &\cong H^*(Z) \\ \delta + \alpha E &\mapsto \rho^*(\delta) + j_*\pi^*(\alpha). \end{aligned} \quad (21)$$

Under the identification of $H^*(Z)$ with $H^*(Y) \oplus H^{*-2}(S)\langle E \rangle$ by means of (21), the cup product of Z is given by

$$\begin{aligned} (\delta_1 + \alpha_1 E) \cdot (\delta_2 + \alpha_2 E) &= \delta_1 \cup \delta_2 - 2i_*(\alpha_1 \cup \alpha_2) \\ &+ (i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2) + 2R^1\pi_*N_{E/Z} \cup \alpha_1 \cup \alpha_2) E. \end{aligned}$$

Proof. The map (21) is clearly an isomorphism of complex vector spaces. From the projection formula we get

$$j_*\pi^*(\alpha) \cup \rho^*(\delta) = j_*(\pi^*(\alpha) \cdot j^*\rho^*(\delta)) = j_*\pi^*(\alpha \cdot i^*\delta).$$

Hence $\alpha E \cdot \delta = (\alpha \cup i^*\delta)E$. For $\alpha_1, \alpha_2 \in H^*(S)$

$$j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2) = \rho^*(\delta) + j_*\pi^*(\alpha)$$

for some $\delta \in H^*(Y)$ and $\alpha \in H^*(S)$. Using again the projection formula we have

$$j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2) = j_*(N_{E/Z} \cup \pi^*(\alpha_1 \cup \alpha_2)).$$

Therefore

$$\delta = \rho_*(j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2)) = -2i_*(\alpha_1 \cup \alpha_2).$$

To determine α we notice that $\pi^*(\alpha)$ is the coefficient of $\mathcal{O}_F(-2)$ in $j^*(j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2))$, hence

$$\alpha = 2\alpha_1 \cup \alpha_2 \cup R^1\pi_*N_{E/Z}.$$

□

The A_n -case

Proposition 4.9. *Let Y be a variety with transversal A_n -singularities whose associated orbifold $[Y]$ has trivial monodromy. Then the map below is an isomorphism of vector spaces*

$$\begin{aligned} H^*(Y) \oplus_{l=1}^n H^{*-2}(S)\langle E_l \rangle &\rightarrow H^*(Z) \\ \delta + \alpha_1 E_1 + \dots + \alpha_n E_n &\mapsto \rho^*(\delta) + \sum_{l=1}^n j_{l*}\pi_l^*(\alpha_l). \end{aligned} \quad (22)$$

Under this identification the cup product of Z is given by

$$E_i \cup E_j = \rho_*(E_i \cup E_j) + \sum_{l=1}^n \alpha_l E_l, \quad (23)$$

where the vector $(\alpha_1, \dots, \alpha_n)$ is

- $(0, \dots, 0)$ if $|i - j| > 1$;
- if $i = j - 1$ and $j \in \{2, \dots, n\}$ it is defined by the system

$$\begin{pmatrix} 0 \\ \dots \\ 0 \\ jK - M \\ M - (j-1)K \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \dots \\ \dots \\ \alpha_{j-1} \\ \alpha_j \\ \dots \\ \dots \\ \alpha_n \end{pmatrix};$$

- if $i = j$ and $j \in \{1, \dots, n\}$ then it is defined by

$$\begin{pmatrix} 0 \\ \dots \\ 0 \\ M - (j-1)K \\ -4K \\ (j+1)K - M \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \dots \\ \dots \\ \alpha_{j-1} \\ \alpha_j \\ \alpha_{j+1} \\ \dots \\ \dots \\ \alpha_n \end{pmatrix}.$$

For the proof we need two lemmas:

Lemma 4.10. *For any q we have an exact sequence*

$$0 \rightarrow H^q(Y) \xrightarrow{\rho^*} H^q(Z) \xrightarrow{[j^*]} H^q(E)/\pi^*(H^q(S)) \rightarrow 0,$$

where $[j^*]$ is the composition of j^* with the projection $H^q(E) \rightarrow H^q(E)/\pi^*(H^q(S))$. The sequence splits, so we get an isomorphism of vector spaces

$$H^q(Z) \cong H^q(Y) \oplus H^q(E)/\pi^*(H^q(S)).$$

Proof. The exactness follows by comparing the exact sequences of the pairs (E, Z) and (S, Y) . The sequence splits since there exists a push-forward morphism $\rho_* : H^*(Z) \rightarrow H^*(Y)$ which satisfies $\rho_* \circ \rho^* = id_{H^*(Y)}$. \square

Lemma 4.11. *There is a canonical isomorphism of vector spaces*

$$H^*(E)/\pi^*(H^*(S)) \cong \oplus_{l=1}^n H^*(E_l)/\pi_l^*(H^*(S)).$$

Proof. This is an easy consequence of the structure of the cohomology of \mathbb{P}^1 -bundles. \square

Proof of Proposition 4.9. Let us denote by c_n the $n \times n$ matrix which is minus the Cartan matrix,

$$c_n = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \quad (24)$$

As a consequence of the above Lemmas we have that the vector spaces $H^*(Y) \oplus_{l=1}^n H^{*-2}(S)\langle E_l \rangle$ and $H^*(Z)$ have the same dimension, so it is enough to show that the map in (22) is injective. Hence let us assume that

$$\rho^*(\delta) + \sum_{l=1}^n j_{l*} \pi_l^*(\alpha_l) = 0. \quad (25)$$

Then $\delta = \rho_*(\rho^*(\delta) + \sum_{l=1}^n j_{l*} \pi_l^*(\alpha_l)) = 0$. Moreover, applying j_k^* to (25) we get the following equation up to elements in $\pi_k^*(H^*(S))$,

$$\begin{aligned} 0 &= j_k^* \left(\sum_{l=1}^n j_{l*} \pi_l^*(\alpha_l) \right) \\ &= \pi_k^*(\alpha_{k-1})[E_{k-1} \cap E_k \subset E_k] + \pi_k^*(\alpha_k)N_{E_k/Z} + \pi_k^*(\alpha_{k+1})[E_{k+1} \cap E_k \subset E_k] \\ &= \pi_k^*(\alpha_{k-1} - 2\alpha_k + \alpha_{k+1})\mathcal{O}_{F_k}(1), \end{aligned} \quad (26)$$

where, by $[E_{k-1} \cap E_k \subset E_k]$ (resp. $[E_{k+1} \cap E_k \subset E_k]$) we mean the cohomology class dual to the homology class of $E_{k-1} \cap E_k$ (resp. $E_{k+1} \cap E_k$) in E_k . Equation (26) is a consequence of Prop. 4.6 and the following identities (see [20]):

$$\begin{aligned} [E_{l-1} \cap E_l \subset E_{l-1}] &= c_1(\mathcal{O}_{F_{l-1}}(1) \otimes \pi_{l-1}^* L_{l-1}) \\ [E_{l-1} \cap E_l \subset E_l] &= c_1(\mathcal{O}_{F_l}(1) \otimes \pi_l^* M_l). \end{aligned}$$

From (26) we have $(c_n)_{kl}\alpha_l = 0$ for any $k = 1, \dots, n$. Since c_n is non-degenerate, $\alpha_l = 0$ for all l . This shows that the map is injective and hence an isomorphism.

Then to prove (23), we write

$$E_i \cup E_j = \rho^*(\delta) + \sum_{l=1}^n j_{l*} \pi_l^*(\alpha_l), \quad (27)$$

where $\delta \in H^*(Y)$ and $\alpha_l \in H^*(S)$. Then

$$\delta = \rho_*(E_i \cup E_j) = \begin{cases} 0 & \text{if } |i - j| > 1 \\ [S] & \text{if } |i - j| = 1 \\ -2[S] & \text{if } |i - j| = 0. \end{cases}$$

To determine the α_l 's, we pull-back through j_k both sides of (27) obtaining, up to elements in $\pi_k^*(H^*(S))$,

$$j_k^*(E_i \cup E_j) = \pi_k^*(\alpha_{k-1} - 2\alpha_k + \alpha_{k+1})\mathcal{O}_{F_k}(1).$$

On the other hand, the left side of (27) reads

$$j_k^*(E_i \cup E_j) = j_k^*(j_{i*}([E_i]) \cup j_{j*}([E_j])) = [E_i \cap E_k \subset E_k] \cup [E_j \cap E_k \subset E_k]. \quad (28)$$

We now distinguish three cases.

Case $|i - j| > 1$. Then $E_i \cup E_j = 0$.

Case $i = j - 1$, $j \in 2, \dots, n$. Then

$$j_k^*(E_{j-1} \cup E_j) = \begin{cases} 0 & \text{for } k < j - 1, \\ N_{E_{j-1}/Z} \cup [E_{j-1} \cap E_j \subset E_{j-1}] & \text{for } k = j - 1, \\ N_{E_j/Z} \cup [E_{j-1} \cap E_j \subset E_j] & \text{for } k = j, \\ 0 & \text{for } k > j. \end{cases}$$

In order to compute $N_{E_l/Z}$ we proceed as in the last part of the proof of Prop. 4.6. We get

$$N_{E_l/Z} \cong \mathcal{O}_{F_l}(-2) + \pi_l^*(K - L_l - M_l). \quad (29)$$

Therefore, up to elements in $\pi_k^*(H^*(S))$, we have

$$j_k^*(E_{j-1} \cup E_j) = \begin{cases} 0 & \text{for } k < j - 1, \\ \mathcal{O}_{F_{j-1}}(1)(jK - M) & \text{for } k = j - 1, \\ \mathcal{O}_{F_j}(1)(M - (j - 1)K) & \text{for } k = j, \\ 0 & \text{for } k > j. \end{cases}$$

Then the α_l 's are uniquely determined by the following system

$$\begin{pmatrix} 0 \\ \dots \\ 0 \\ jK - M \\ M - (j - 1)K \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \dots \\ \dots \\ \alpha_{j-1} \\ \alpha_j \\ \dots \\ \dots \\ \alpha_n \end{pmatrix}$$

Case $|i - j| = 0$. This case is analogous to the previous one hence we omit the computations. \square

5 Quantum corrections

In this Section we compute the quantum corrected cohomology ring (as introduced in Def. 1.8) of the crepant resolution $\rho : Z \rightarrow Y$ of a variety with transversal A_n singularities. We will assume that the orbifold $[Y]$ associated to Y has trivial monodromy.

5.1 Gromov-Witten invariants of the crepant resolution

We give here a conjectural formula for the genus zero Gromov-Witten invariants of Z which are needed to compute the quantum corrected cohomology ring. We will use Notation 4.5. Moreover we identify $H^*(Z)$ with $H^*(Y) \oplus_{i=1}^n H^{*-2}(S)\langle E_i \rangle$ by means of the isomorphism (22), so that a cohomology class $\gamma \in H^*(Z)$ of Z will be denoted by

$$\gamma = \delta + \alpha_1 E_1 + \dots + \alpha_n E_n, \quad \text{with } \delta \in H^*(Y), \alpha_i \in H^{*-2}(S).$$

Let $\beta_i \in H_2(Z, \mathbb{Z})$ be the class of a fiber of $\pi_i : E_i \rightarrow S$. Then β_1, \dots, β_n is an integral basis of $\text{Ker } \rho_*$ (see Assumption 1.3).

We will denote by $\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3)$ the genus zero Gromov-Witten invariant of Z and homology class $\Gamma = a_1 \beta_1 + \dots + a_n \beta_n \in H_2(Z, \mathbb{Z})$, namely

$$\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) = \int_{[\bar{\mathcal{M}}_{0,3}(Z, \Gamma)]^{\text{vir}}} ev_3^*(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \quad (30)$$

where $\gamma_i \in H^*(Z)$, $\Gamma \in \text{Ker } \rho_*$, $\bar{\mathcal{M}}_{0,3}(Z, \Gamma)$ is the moduli space of 3-pointed stable maps $[\mu : (C, p_1, p_2, p_3) \rightarrow Z]$ such that $\mu_*[C] = \Gamma$, the arithmetic genus of C is 0, and $ev_3 : \bar{\mathcal{M}}_{0,3}(Z, \Gamma) \rightarrow Z \times Z \times Z$ is the evaluation map.

Conjecture 5.1. *Under the previous hypothesis, the following expression holds for the Gromov-Witten invariants:*

$$\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) = \begin{cases} 0 & \text{if } \gamma_1, \gamma_2 \text{ or } \gamma_3 \text{ are in } H^*(Y); \\ (E_{l_1} \cdot \beta_{\mu\nu})(E_{l_2} \cdot \beta_{\mu\nu})(E_{l_3} \cdot \beta_{\mu\nu}) \int_S \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot R^1 \pi_* N_{E/Z} & \text{if } \Gamma = \beta_{\mu\nu} \text{ and } \gamma_i = \alpha_i \cdot E_{l_i} \text{ for } i \in \{1, 2, 3\}; \\ 0 & \text{in the remaining cases.} \end{cases}$$

where the second possibility holds if $\Gamma = a \cdot \beta_{\mu\nu}$ with $\beta_{\mu\nu} := \beta_\mu + \dots + \beta_\nu$ for $\mu, \nu \in \{1, \dots, n\}$ with $\mu \leq \nu$, and $\gamma_i = \alpha_i \cdot E_{l_i}$ for $i \in \{1, 2, 3\}$.

Remark 5.2. We report here an outline of the proof of Conjecture 5.1. The complete proof is given in a work in progress with B. Fantechi where we also compute the quantum corrections in the transversal D and E cases. In Section 7 we prove this conjecture in the A_1 -case, in the A_n -case if $\Gamma = \beta_{\mu\nu}$, and also in the A_n -case for any Γ under some additional hypothesis on Z . These results will be used in order to prove the conjecture in the general case.

Our references for virtual fundamental classes are [4] and [30]. In particular notations are taken from [4].

It follows from Lem. 7.1 the existence of a morphism

$$\phi : \bar{\mathcal{M}}_{0,0}(Z, \Gamma) \rightarrow S \quad (31)$$

of Deligne-Mumford stacks such that, if $\Gamma = \beta_{\mu\nu}$, then it is an isomorphism. Under the identification of $\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})$ with S by means of (31), we have, by Th. 7.9,

$$[\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})]^{\text{vir}} = c_1(R^1 \pi_* N_{E/Z}).$$

Therefore, Conj. 5.1 is equivalent to the following statement (see Lem. 7.5):

$$\phi_*[\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{\text{vir}} = a[\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})]^{\text{vir}} \quad (32)$$

where

$$a = \begin{cases} \frac{1}{d^3} & \text{if } \Gamma = d\beta_{\mu\nu} \\ 0 & \text{otherwise.} \end{cases} \quad (33)$$

Notice that $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)$ and $\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})$ have the same virtual dimension.

Let $E_\Gamma^\bullet \rightarrow L_{\bar{\mathcal{M}}_{0,0}(Z, \Gamma)}^\bullet$ and $E_{\beta_{\mu\nu}}^\bullet \rightarrow L_{\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})}^\bullet$ denote the standard obstruction theories of Gromov-Witten theory. There exists a morphism

$$\Theta : \phi^* E_{\beta_{\mu\nu}}^\bullet \rightarrow E_\Gamma^\bullet$$

in the derived category $D(\mathcal{O}_{\bar{\mathcal{M}}_{0,0}(Z, \Gamma)_{\text{ét}}})$. Let $C^\bullet(\Theta)$ be the mapping cone of Θ , then by standard properties of the mapping cone we have the commutative diagram below

$$\begin{array}{ccccc} \phi^* E_{\beta_{\mu\nu}}^\bullet & \xrightarrow{\Theta} & E_\Gamma^\bullet & \longrightarrow & C^\bullet(\Theta) \xrightarrow{+1} \\ \downarrow & & \downarrow & & \downarrow \\ \phi^* L_{\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})}^\bullet & \longrightarrow & L_{\bar{\mathcal{M}}_{0,0}(Z, \Gamma)}^\bullet & \longrightarrow & L_\phi^\bullet \xrightarrow{+1} \end{array} \quad (34)$$

where the rows are distinguished triangles and L_ϕ^\bullet denotes the relative cotangent complex of ϕ . It turns out that $C^\bullet(\Theta) \rightarrow L_\phi^\bullet$ is a relative perfect obstruction theory and its restriction on each fiber of ϕ is an obstruction theory of virtual dimension 0. As a consequence we have (32). To determine the constant a in (32) we can assume that Z satisfies the hypothesis of Th. 7.15. This proves (33) which complete the proof.

Remark 5.3. Notice that, if $[Y]$ carries a global holomorphic symplectic 2-form ω , then we can identify L with M^\vee by means of ω . Hence

$$(n+1)K \cong M \otimes L \cong \mathcal{O}_S$$

and all the Gromov-Witten invariants vanish.

5.2 Quantum corrected cohomology ring

Proposition 5.4. *Let Y be a variety with transversal A_n singularities such that $n = 1$ or $n \geq 2$ and the corresponding orbifold $[Y]$ has trivial monodromy. Let $\rho : Z \rightarrow Y$ be the crepant resolution. Then, with the hypothesis under which Conj. 5.1 holds, the quantum corrected cup product $*_\rho$ is given by*

$$E_i *_\rho E_j = \rho_*(E_i \cup E_j) + \sum_{l,m=1}^n (c_n^{-1})_{lm} \{R_{ijm}(\underline{q}) R^1 \pi_* N_{E/Z} + \alpha_{ijm}\} E_l, \quad (35)$$

where (c_n^{-1}) is the inverse matrix of (24), $\underline{q} := (q_1, \dots, q_n)$,

$$R_{ijm}(\underline{q}) = \sum_{1 \leq \mu \leq \nu \leq n} (E_i \cdot \beta_{\mu\nu})(E_j \cdot \beta_{\mu\nu})(E_m \cdot \beta_{\mu\nu}) \frac{q_\mu \cdots q_\nu}{1 - q_\mu \cdots q_\nu}.$$

Here $\beta_{\mu\nu} := \beta_\mu + \dots + \beta_\nu$ and $\alpha_{ijm} = \alpha = -4R^1 \pi_* N_{E/Z}$ if $n = 1$, otherwise it is defined by

$$E_i \cup E_j = \rho_*(E_i \cup E_j) + \sum_{l,m=1}^n (c_n^{-1})_{lm} \alpha_{ijm} E_l$$

(see Prop. 4.9).

Proof. First of all we notice that if $\gamma_1 \in H^*(Y)$ or $\gamma_2 \in H^*(Y)$, then $\gamma_1 *_\rho \gamma_2 = \gamma_1 \cup \gamma_2$. Indeed, in this case, all the Gromov-Witten invariants $\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3)$ are 0 if $\Gamma \in \text{Ker} \rho_*$. Therefore the quantum corrected cohomology ring is determined by $E_i *_\rho E_j$, for all $i, j \in \{1, \dots, n\}$.

By definition we have that

$$E_i *_\rho E_j = E_i \cup E_j + E_i \cup_{qc} E_j,$$

where $E_i \cup_{qc} E_j$ is defined by the equations

$$\langle E_i \cup_{qc} E_j, \gamma \rangle = \langle E_i, E_j, \gamma \rangle_{qc}(q_1, \dots, q_n), \quad \gamma \in H^*(Z). \quad (36)$$

From Prop. 4.9, it is enough to show that

$$E_i \cup_{qc} E_j = \sum_{l,m=1}^n (c_n^{-1})_{lm} R_{ijm}(\underline{q}) R^1 \pi_* N_{E/Z} E_l. \quad (37)$$

In general, we have

$$E_i \cup_{qc} E_j = \epsilon_1(\underline{q}) E_1 + \dots + \epsilon_n(\underline{q}) E_n$$

for some $\epsilon_1(\underline{q}), \dots, \epsilon_n(\underline{q}) \in H^*(S)$. To lighten the notations, we will denote $\epsilon_1(\underline{q})$ with ϵ_l for all $l \in \{1, \dots, n\}$. Notice that $E_i \cup_{qc} E_j \in H^*(Y)^\perp$, where $H^*(Y)^\perp$ is the subspace of $H^*(Z)$ which is orthogonal to $H^*(Y)$ with respect to the Poincaré pairing. We compute the left hand side of (36):

$$\begin{aligned} \langle E_i \cup_{qc} E_j, \alpha E_k \rangle &= \int_Z \sum_{l=1}^n j_{l*} \pi_l^*(\epsilon_l) \cup \alpha E_k \\ &= \sum_{l=1}^n \int_Y \rho_*(j_{l*} \pi_l^*(\epsilon_l) \cup \alpha E_k) \\ &= \sum_{l=1}^n \int_Y \rho_* j_{l*} (\pi_l^*(\epsilon_l) \cup j_l^*(\alpha E_k)) \\ &= \sum_{l=1}^n \int_Y i_* \pi_{l*} (\pi_l^*(\epsilon_l \cup \alpha) \cup [E_l \cap E_k \subset E_l]) \\ &= \int_S (\epsilon_{k-1} - 2\epsilon_k + \epsilon_{k+1}) \cup \alpha. \end{aligned}$$

On the other hand, the right hand side of (36) is given by

$$\begin{aligned}\langle E_i, E_j, \alpha E_k \rangle_{qc}(\underline{q}) &= \sum_{a=1}^{\infty} \sum_{1 \leq \mu \leq \nu \leq n} (q_\mu \cdots q_\nu)^a (E_i \cdot \beta_{\mu\nu})(E_j \cdot \beta_{\mu\nu})(E_k \cdot \beta_{\mu\nu}) \int_S \alpha R^1 \pi_* N_{E/Z} \\ &= R_{ijk}(\underline{q}) \int_S \alpha R^1 \pi_* N_{E/Z}.\end{aligned}$$

Therefore, comparing the two expressions, (37) holds. \square

We give now a description of the line bundle $R^1 \pi_* N_{E/Z}$. This will be used to compare the Chen-Ruan cohomology of $[Y]$ and the quantum corrected cohomology of Z .

Lemma 5.5. *For any $\mu \leq \nu$, $\mu, \nu \in 1, \dots, n$, there is an isomorphism*

$$R^1 \pi_{\mu\nu *} N_{E_{\mu\nu}/Z} \cong K$$

where K is the line bundle defined in Not. 3.10.

Proof. Let us first assume that $\mu < \nu$. Then let a be an integer which satisfies $i \leq a < j$, $\tilde{E}_a := E_i \cup \dots \cup E_a$ and $\tilde{E}_{a+1} := E_{a+1} \cup \dots \cup E_j$. Let us denote $S_a = \tilde{E}_a \cap \tilde{E}_{a+1}$. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{E_{\mu\nu}} \rightarrow \mathcal{O}_{\tilde{E}_a} \oplus \mathcal{O}_{\tilde{E}_{a+1}} \rightarrow \mathcal{O}_{S_a} \rightarrow 0.$$

This gives the exact sequence

$$0 \rightarrow N_{E_{\mu\nu}/Z} \rightarrow N_{\tilde{E}_a/Z}(S_a) \oplus N_{\tilde{E}_{a+1}/Z}(S_a) \rightarrow N_{E_{\mu\nu}/Z}|_{S_a} \rightarrow 0. \quad (38)$$

We now claim that

$$R^1 \pi_{\mu\nu *} N_{E_{\mu\nu}/Z} \cong N_{E/Z}|_{S_a}.$$

This follows from the long exact sequence which is obtained applying the functor $R^\bullet \pi_*$ to (38). Notice that

$$R^p \pi_* N_{\tilde{E}_a/Z}(S_a) = R^p \pi_* N_{\tilde{E}_{a+1}/Z}(S_a) = 0 \quad \text{for all } p \geq 0. \quad (39)$$

Then, since

$$N_{E/Z}|_{S_a} \cong N_{S_a/\tilde{E}_a} \otimes N_{S_a/\tilde{E}_{a+1}}$$

we get the result by considering the explicit description of the divisors E_i in terms of the line bundles L_i, M_i and K given in Prop. 4.6.

If $\mu = \nu = l$, then the result follows from (29). \square

6 Verification of Conjecture 1.9 for A_1 and A_2 singularities

We put together the computations of the previous Sections in order to verify Conj. 1.9 in the A_1 and A_2 -case.

6.1 The A_1 -case

In this case Conj. 5.1 is proved in Th. 7.6, so the quantum corrected cohomology ring reads

$$\begin{aligned}(\delta_1 + \alpha_1 E) *_{\rho} (\delta_2 + \alpha_2 E) &= \delta_1 \cup \delta_2 - 2i_*(\alpha_1 \cup \alpha_2) \\ &\quad + (i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2)) E \\ &\quad + \left((2 + 4\frac{q}{1-q}) R^1 \pi_* N_{E/Z} \cup \alpha_1 \cup \alpha_2 \right) E.\end{aligned}$$

On the other hand, the orbifold cup product is given by

$$\begin{aligned}(\delta_1 + \alpha_1 e) \cup_{\text{CR}} (\delta_2 + \alpha_2 e) &= \delta_1 \cup \delta_2 + \frac{1}{2} i_*(\alpha_1 \cup \alpha_2) \\ &\quad + (i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2)) e.\end{aligned}$$

It is easy to see that the morphism below is a ring isomorphism

$$\begin{aligned}H_{\text{CR}}^*([Y]) &\rightarrow H^*(Z)(-1) \\ (\delta, \alpha) &\mapsto (\delta, \frac{\sqrt{-1}}{2} \alpha).\end{aligned}$$

6.2 The A_2 -case

Here $[Y]$ denotes an orbifold with transversal A_2 singularities and trivial monodromy, $\rho : Z \rightarrow Y$ is the crepant resolution. We assume that Z satisfies the hypothesis under which Conj. 5.1 holds.

Notation 6.1. We define $\delta_1 := \frac{q_1}{1-q_1}$, $\delta_2 := \frac{q_2}{1-q_2}$ and $\delta_3 := \frac{q_1 q_2}{1-q_1 q_2}$.

In this case, the quantum corrected cohomology ring $H_\rho^*(Z)(q_1, q_2)$ can be given explicitly using Prop. 5.4:

$$\begin{aligned} E_1 *_\rho E_1 &= -2[S] + \frac{1}{3} [(4\delta_1 + \delta_3 + 2)L + (4\delta_1 + \delta_3 + 3)M] E_1 \\ &\quad + \frac{1}{3} [(\delta_2 + \delta_3)L + (\delta_2 + \delta_3 + 2)M] E_2 \\ E_1 *_\rho E_2 &= [S] + \frac{1}{3} [(-2\delta_1 + \delta_3 - 1)L + (-2\delta_1 + \delta_3)M] E_1 \\ &\quad + \frac{1}{3} [(-2\delta_2 + \delta_3)L + (-2\delta_2 + \delta_3 - 1)M] E_2 \\ E_2 *_\rho E_2 &= -2[S] + \frac{1}{3} [(\delta_1 + \delta_3 + 2)L + (\delta_1 + \delta_3)M] E_1 \\ &\quad + \frac{1}{3} [(4\delta_2 + \delta_3 + 3)L + (4\delta_2 + \delta_3 + 2)M] E_2. \end{aligned}$$

On the other hand, the Chen-Ruan cohomology ring $H_{\text{CR}}^*([Y])$ has the expression (Th. 3.12, see also Ex. 3.15)

$$\begin{aligned} e_1 \cup_{\text{CR}} e_1 &= \frac{1}{3} L e_2 \\ e_1 \cup_{\text{CR}} e_2 &= \frac{1}{3} [S] \\ e_2 \cup_{\text{CR}} e_2 &= \frac{1}{3} M e_1. \end{aligned}$$

We look for a linear map

$$\begin{aligned} H_\rho^*(Z)(q_1, q_2) &\rightarrow H_{\text{CR}}^*([Y]) \\ E_1 &\mapsto a e_1 + b e_2 \\ E_2 &\mapsto c e_1 + d e_2 \end{aligned} \tag{40}$$

and (q_1, q_2) such that (40) is a ring isomorphism.

First of all we notice that the previous expressions for the quantum corrected cup product $*_\rho$ and for the orbifold cup product \cup_{CR} are symmetric if we exchange E_1 with E_2 , L with M and e_1 with e_2 . So we impose the conditions $b = c$ and $a = d$. In order that (40) is a ring isomorphism, a, b, q_1 and q_2 have to satisfy the equations

$$\begin{aligned} \frac{2}{3} ab[S] + \frac{1}{3} b^2 M e_1 + \frac{1}{3} a^2 L e_2 &= -2[S] + \frac{1}{3} [(4\delta_1 + \delta_3 + 2)L + (4\delta_1 + \delta_3 + 3)M] (a e_1 + b e_2) \\ &\quad + \frac{1}{3} [(\delta_2 + \delta_3)L + (\delta_2 + \delta_3 + 2)M] (b e_1 + a e_2), \\ \frac{a^2 + b^2}{3} [S] + \frac{ab}{3} M e_1 + \frac{ab}{3} L e_2 &= [S] + \frac{1}{3} [(-2\delta_1 + \delta_3 - 1)L + (-2\delta_1 + \delta_3)M] (a e_1 + b e_2) \\ &\quad + \frac{1}{3} [(-2\delta_2 + \delta_3)L + (-2\delta_2 + \delta_3 - 1)M] (b e_1 + a e_2). \end{aligned}$$

Now, identifying the coefficients of e_1 and e_2 , we get the following system of equations:

$$\begin{aligned} \frac{2}{3} ab &= -2 \\ \frac{a^2 + b^2}{3} &= 1 \\ a(4\delta_1 + \delta_3 + 2)L + a(4\delta_1 + \delta_3 + 3)M + b(\delta_2 + \delta_3)L + b(\delta_2 + \delta_3 + 2)M &= b^2 M \\ b(4\delta_1 + \delta_3 + 2)L + b(4\delta_1 + \delta_3 + 3)M + a(\delta_2 + \delta_3)L + a(\delta_2 + \delta_3 + 2)M &= a^2 L \\ a(-2\delta_1 + \delta_3 - 1)L + a(-2\delta_1 + \delta_3)M + b(-2\delta_2 + \delta_3)L + b(-2\delta_2 + \delta_3 - 1)M &= abM \\ b(-2\delta_1 + \delta_3 - 1)L + b(-2\delta_1 + \delta_3)M + a(-2\delta_2 + \delta_3)L + a(-2\delta_2 + \delta_3 - 1)M &= abL. \end{aligned}$$

In special cases there could be relations between L and M so that the previous system can be simplified, but in general they are independent. The resulting system can be solved and the solutions are

$$(a, b, q_1, q_2) = (\sqrt{3}\exp(\frac{7}{6}\pi i), \sqrt{3}\exp(\frac{11}{6}\pi i), \exp(\frac{2}{3}\pi i), \exp(\frac{2}{3}\pi i)) \quad \text{and} \\ (\sqrt{3}\exp(\frac{5}{6}\pi i), \sqrt{3}\exp(\frac{1}{6}\pi i), \exp(\frac{4}{3}\pi i), \exp(\frac{4}{3}\pi i)).$$

So, we have proved the following.

Proposition 6.2. *Under the hypothesis in the beginning of Section 6.2, if $q_1 = q_2 = \exp(\frac{2}{3}\pi i)$ or $q_1 = q_2 = \exp(\frac{4}{3}\pi i)$, then, the ring $H^*(Z)(q_1, q_2)$ is isomorphic to the Chen-Ruan cohomology ring $H_{\text{CR}}^*([Y])$. Moreover there is a unique isomorphism given by the following linear transformation*

$$\begin{aligned} H_\rho^*(Z)(q_1, q_2) &\rightarrow H_{\text{CR}}^*([Y]) \\ E_1 &\mapsto ae_1 + be_2 \\ E_2 &\mapsto be_1 + ae_2 \end{aligned} \tag{41}$$

where (a, b) is equal to $(\sqrt{3}\exp(\frac{7}{6}\pi i), \sqrt{3}\exp(\frac{11}{6}\pi i))$ in the first case and to $(\sqrt{3}\exp(\frac{5}{6}\pi i), \sqrt{3}\exp(\frac{1}{6}\pi i))$ in the second one.

7 On the Conjecture 5.1

In this section we prove Conj. 5.1 in some cases. As we saw in Rem. 5.2, the fact that it holds in the following cases will be used to prove it in general. First we give some general results which allow us to simplify the computation.

Lemma 7.1. *Under the same hypothesis of Conj. 5.1, there is a morphism*

$$\phi : \bar{\mathcal{M}}_{0,0}(Z, \Gamma) \rightarrow S$$

such that, for any point $p \in S$, the fiber $\phi^{-1}(p)$ is isomorphic to $\bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma)$ (see Not. 2.3). Moreover, there is a covering $U \rightarrow S$ in the complex topology and a Cartesian diagram

$$\begin{array}{ccc} U \times \bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma) & \longrightarrow & \bar{\mathcal{M}}_{0,0}(Z, \Gamma) \\ \text{pr}_1 \downarrow & & \downarrow \phi \\ U & \longrightarrow & S. \end{array} \tag{42}$$

If $\Gamma = \beta_{\mu\nu}$ for $\mu \leq \nu$, then ϕ is an isomorphism.

Proof. *Step 1.* We first prove that for any scheme B of finite type over \mathbb{C} and any object

$$\begin{array}{ccc} C & \xrightarrow{f} & Z \\ p \downarrow & & \\ B & & \end{array}$$

in $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)(B)$, there is a morphism $g : C \rightarrow E$ such that $f = j \circ g$, where $j : E \rightarrow Z$ is the inclusion map.

Let $\mathcal{O}_Z(E)$ be the line bundle over Z associated to the divisor E , and let s be the section of $\mathcal{O}_Z(E)$ defined by E , that is, $s = \{s_i\}$ where s_i are local equations for the Cartier divisor E . Then f factors through E if and only if f^*s vanishes as section of $f^*\mathcal{O}_Z(E)$. We show that $p_*f^*\mathcal{O}_Z(E)$ is the zero sheaf.

First of all we assume that $B = \text{Spec}(\mathbb{C})$. Then

$$\rho_*f_*([C]) = 0,$$

where ρ_* and f_* are the morphisms of Chow groups induced by ρ and f respectively and $[C]$ is the fundamental class of C . It follows that the image of $\rho \circ f$ is a point $y \in S$, so that $f(C) \subset E$. Since C is reduced, f factors through E [23]. Notice that $f(C)$ is contained in a fiber of $\pi : E \rightarrow S$.

Assume now that B is a scheme of finite type over \mathbb{C} and $f : C \rightarrow Z$ is a stable map over B . Given a point $b \in B$, let X be the sub-variety of B whose generic point is b , namely $X = \overline{\{b\}}$. For any closed point $x \in X$ we have that $H^0(C_x, f^*\mathcal{O}_Z(E)|_{C_x}) = 0$ because $f|_{C_x}$ factors through a fiber

of E over S . By Cohomology and Base Change it follows that $(p_* f^* \mathcal{O}_Z(E))_x \otimes k(x) = 0$. Since $p_* f^* \mathcal{O}_Z(E)$ is coherent, it is zero on a neighborhood of x , so it is zero on b .

Step 2. Let $\varphi := \pi \circ g : C \rightarrow S$. We prove that there exists a morphism $\phi : B \rightarrow S$ such that $\varphi = \phi \circ p$.

First of all we define a continuous map $\phi : B \rightarrow S$ such that $\varphi = \phi \circ p$. From *Step 1* we have that, if $b \in B$ is a closed point, we can define $\phi(b)$ by

$$\phi(b) := \pi(f(C_b)).$$

Now, let $b \in B$ be any point, and let X be the sub-variety whose generic point is b . Then we define $\phi(b)$ to be the generic point of the closure of $\phi(X)$ in S . The condition $\varphi = \phi \circ p$ implies that ϕ is continuous. In order to give a morphism $\phi : B \rightarrow S$ it remains to find a morphism of sheaves

$$\phi^\sharp : \mathcal{O}_S \rightarrow \phi_* \mathcal{O}_B.$$

For this, we take the composition of $\varphi^\sharp : \mathcal{O}_S \rightarrow \varphi_* \mathcal{O}_C$ with the canonical isomorphism $\varphi_* \mathcal{O}_C \rightarrow \phi_* \mathcal{O}_B$. Notice that, since $p : C \rightarrow B$ is a flat family of genus zero curves, the canonical morphism $\mathcal{O}_B \rightarrow p_* \mathcal{O}_C$ is an isomorphism.

The existence of an open covering $U \rightarrow S$ such that (42) is Cartesian follows from the local structure of Z .

Finally the last statement follows from the fact that if $\Gamma = \beta_1 + \dots + \beta_n$, then we have an inverse of ϕ . It is given by sending any morphism $B \rightarrow S$ to the stable map

$$\begin{array}{ccc} B \times_S E & \xrightarrow{j \circ pr_2} & Z \\ pr_1 \downarrow & & \\ B & & \end{array}$$

□

Remark 7.2. Notice that, if

$$\begin{array}{ccc} C & \xrightarrow{f} & Z \\ p \downarrow & & \\ B & & \end{array}$$

is an element in $\bar{\mathcal{M}}_{0,0}(Z, \beta_\mu + \dots + \beta_\nu)(B)$, then for any \mathbb{C} -valued point $b \in B$, the morphism $f_b : C_b \rightarrow Z$ is an embedding. Then there is a neighborhood $U \subset B$ of b such that the restriction

$$f_U : C_U \rightarrow Z$$

is a family of embeddings parametrized by U ([42], Note 3 pag. 222). First order deformations of $E_b = f_b(C_b)$ in Z are parametrized by

$$H^0(E_b, N_{E_b/Z}) \cong H^0(E_b, \pi_b^*(T_{S,b}) \oplus N_{E_b/\bar{R}}) = T_{S,b}.$$

This identifies the tangent space of $\bar{\mathcal{M}}_{0,0}(Z, \beta_\mu + \dots + \beta_\nu)$ at b with $T_{S, \phi(b)}$.

Remark 7.3. We write explicitly the morphism on tangent spaces

$$T_{\phi, [C]} : T_{\bar{\mathcal{M}}_{0,0}(Z, \Gamma), [C]} \rightarrow T_{S, x}, \quad (43)$$

where $[C]$ denotes the \mathbb{C} -valued point

$$\begin{array}{ccc} C & \xrightarrow{f} & Z \\ p \downarrow & & \\ \text{Spec}(\mathbb{C}) & & \end{array}$$

in $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)$, and $x = \rho(f(C)) \in S$.

For any \mathbb{C} -module N , the tangent space $T_{\bar{\mathcal{M}}_{0,0}(Z, \Gamma), [C]}(N)$ can be identified with the module

$$\text{Ext}_{\mathbb{C}}^1([f^* \Omega_Z \rightarrow \Omega_C], \mathcal{O}_{\mathbb{C}} \otimes_{\mathbb{C}} N), \quad (44)$$

which parametrizes commutative diagrams of the form

$$\begin{array}{ccccccc}
& & f^* \Omega_Z & \xrightarrow{=} & f^* \Omega_Z & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C \otimes_{\mathbb{C}} N & \longrightarrow & \mathcal{B} & \longrightarrow & \Omega_C \longrightarrow 0
\end{array}$$

[30]. In a neighborhood of $f(C)$, Z is isomorphic to $U \times \tilde{R}$, where $U \subset S$ is an open neighborhood of x in S . Hence $f = (f_{\tilde{R}}, f_S)$,

$$f^* \Omega_Z \cong f_{\tilde{R}}^* \Omega_{\tilde{R}} \oplus f_S^* \Omega_{S,x}$$

and the module (44) is isomorphic to

$$\mathrm{Ext}_{\mathbb{C}}^1([f_{\tilde{R}}^* \Omega_{\tilde{R}} \rightarrow \Omega_C], \mathcal{O}_C \otimes_{\mathbb{C}} N) \times_{\mathrm{Ext}_{\mathbb{C}}^1(\Omega_C, \mathcal{O}_C \otimes_{\mathbb{C}} N)} \mathrm{Ext}_{\mathbb{C}}^1([f_S^* \Omega_{S,x} \rightarrow \Omega_C], \mathcal{O}_C \otimes_{\mathbb{C}} N). \quad (45)$$

Since $f_S^* \Omega_{S,x} \rightarrow \Omega_C$ is the zero morphism, we get a morphism

$$\mathrm{Ext}_{\mathbb{C}}^1([f_S^* \Omega_{S,x} \rightarrow \Omega_C], \mathcal{O}_C \otimes_{\mathbb{C}} N) \rightarrow \mathrm{Hom}_{\mathbb{C}}(f_S^* \Omega_{S,x}, \mathcal{O}_C \otimes_{\mathbb{C}} N) \cong T_{S,x} \otimes N. \quad (46)$$

Then (43) is the composition of the projection of (45) on the second factor with (46).

Lemma 7.4. *Let γ_1, γ_2 or γ_3 be elements of $H^*(Y)$. Then*

$$\Psi_{\Gamma}^Z(\gamma_1, \gamma_2, \gamma_3) = 0$$

for any $\Gamma = a_1 \beta_1 + \dots + a_n \beta_n$.

Proof. By the Equivariance Axiom for Gromov-Witten invariants (see e.g. [13]) we can assume that $\gamma_3 = \rho^*(\delta_3)$. The virtual dimension of $\bar{\mathcal{M}}_{0,3}(Z, \Gamma)$ is equal to the dimension of Z . Hence let $\gamma_1, \gamma_2, \gamma_3$ be cohomology classes such that

$$\deg(\gamma_1) + \deg(\gamma_2) + \deg(\delta_3) = \dim Z.$$

We have the following commutative diagram

$$\begin{array}{ccc}
\bar{\mathcal{M}}_{0,3}(Z, \Gamma) & \xrightarrow{ev_3} & Z \times Z \times Z \\
f_{3,2} \times f_{3,0} \downarrow & & \downarrow id \times id \times \rho \\
\bar{\mathcal{M}}_{0,2}(Z, \Gamma) \times \bar{\mathcal{M}}_{0,0}(Z, \Gamma) & \xrightarrow{ev_2 \times \varphi} & Z \times Z \times Y
\end{array}$$

where $\varphi = i \circ \phi$. Then

$$\begin{aligned}
ev_3^*(\gamma_1 \otimes \gamma_2 \otimes \rho^*(\delta_3)) &= (f_{3,2} \times f_{3,0})^*(ev_2 \times \varphi)^*(\gamma_1 \otimes \gamma_2 \otimes \delta_3) \\
&= [f_{3,2}^* ev_2^*(\gamma_1 \otimes \gamma_2)] \cdot [f_{3,0}^* \varphi^*(\delta_3)] \\
&= f_{3,2}^* ([ev_2^*(\gamma_1 \otimes \gamma_2)] \cdot [f_{2,0}^* \varphi^*(\delta_3)])
\end{aligned}$$

where we have used the fact $f_{3,0} = f_{2,0} \circ f_{3,2}$. On the other hand, the following equalities hold

$$\begin{aligned}
[\bar{\mathcal{M}}_{0,3}(Z, \Gamma)]^{vir} &= f_{3,0}^* [\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{vir} \\
&= f_{3,2}^* f_{2,0}^* [\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{vir} \\
&= f_{3,2}^* [\bar{\mathcal{M}}_{0,2}(Z, \Gamma)]^{vir}.
\end{aligned}$$

Then

$$\begin{aligned}
\Psi_{\Gamma}^Z(\gamma_1, \gamma_2, \rho^*(\delta_3)) &= \int_{f_{3,2}^* [\bar{\mathcal{M}}_{0,2}(Z, \Gamma)]^{vir}} f_{3,2}^* ([ev_2^*(\gamma_1 \otimes \gamma_2)] \cdot [f_{2,0}^* \varphi^*(\delta_3)]) \\
&= (\text{constant}) \cdot \int_{[\bar{\mathcal{M}}_{0,2}(Z, \Gamma)]^{vir}} [ev_2^*(\gamma_1 \otimes \gamma_2)] \cdot [f_{2,0}^* \varphi^*(\delta_3)]
\end{aligned}$$

which is zero since the virtual dimension of $\mathcal{M}_{0,2}(Z, \Gamma)$ is the virtual dimension of $\mathcal{M}_{0,3}(Z, \Gamma)$ minus 1. \square

It remains to compute the invariants of the form

$$\Psi_{\Gamma}^Z(j_{l_1*} \pi_{l_1}^*(\alpha_1), j_{l_2*} \pi_{l_2}^*(\alpha_2), j_{l_2*} \pi_{l_2}^*(\alpha_2)),$$

where $\alpha_1, \alpha_2, \alpha_3 \in H^*(S)$ satisfy the equation

$$\deg \alpha_1 + \deg \alpha_2 + \deg \alpha_3 = \dim S - 1.$$

With the next Lemma, we reduce this computation to an integral over the class

$$\phi_*[\mathcal{M}_{0,0}(Z, \Gamma)]^{vir} \in A_{\dim S - 1}(S)$$

Lemma 7.5. *In the above situation the following equality holds,*

$$\begin{aligned} & \Psi_{\Gamma}^Z(j_{l_1*}\pi_{l_1}^*(\alpha_1), j_{l_2*}\pi_{l_2}^*(\alpha_2), j_{l_3*}\pi_{l_3}^*(\alpha_3)) = \\ &= (E_{l_1} \cdot \Gamma)(E_{l_2} \cdot \Gamma)(E_{l_3} \cdot \Gamma) \int_{\phi_*[\mathcal{M}_{0,0}(Z, \Gamma)]^{vir}} (\alpha_1 \cdot \alpha_2 \cdot \alpha_3). \end{aligned}$$

Proof. Consider the Cartesian diagram below which defines $E \times_S E \times_S E$

$$\begin{array}{ccc} E \times_S E \times_S E & \longrightarrow & E \times E \times E \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \times S \times S \end{array}$$

where the arrow in the last line is the diagonal embedding.

From Lemma 7.1 the evaluation morphism $ev_3 : \bar{\mathcal{M}}_{0,3}(Z, \Gamma) \rightarrow Z \times Z \times Z$ factors through a morphism $e\tilde{v}_3 : \bar{\mathcal{M}}_{0,3}(Z, \Gamma) \rightarrow E \times_S E \times_S E$ and the inclusion $E \times_S E \times_S E \rightarrow Z \times Z \times Z$. Then

$$\begin{aligned} & \Psi_{\Gamma}^Z(j_{l_1*}\pi_{l_1}^*(\alpha_1), j_{l_2*}\pi_{l_2}^*(\alpha_2), j_{l_3*}\pi_{l_3}^*(\alpha_3)) = \\ &= \int_{[\bar{\mathcal{M}}_{0,3}(Z, \Gamma)]^{vir}} e\tilde{v}_3(\mathcal{O}_E(E_{l_1}) \otimes \mathcal{O}_E(E_{l_2}) \otimes \mathcal{O}_E(E_{l_3})) \pi^*(\alpha_1 \cdot \alpha_2 \cdot \alpha_3). \end{aligned}$$

We now apply the divisor axiom getting

$$\begin{aligned} & \Psi_{\Gamma}^Z(j_{l_1*}\pi_{l_1}^*(\alpha_1), j_{l_2*}\pi_{l_2}^*(\alpha_2), j_{l_3*}\pi_{l_3}^*(\alpha_3)) = \\ & (E_{l_1} \cdot \Gamma)(E_{l_2} \cdot \Gamma)(E_{l_3} \cdot \Gamma) \int_{[\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{vir}} \phi^*(\alpha_1 \cdot \alpha_2 \cdot \alpha_3) = \end{aligned} \quad (47)$$

$$(E_{l_1} \cdot \Gamma)(E_{l_2} \cdot \Gamma)(E_{l_3} \cdot \Gamma) \int_{\phi_*[\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{vir}} (\alpha_1 \cdot \alpha_2 \cdot \alpha_3). \quad (48)$$

□

7.1 Proof of Conj. 5.1 in the A_1 -case

This case is a generalization of the computation of the Gromov-Witten invariants done in [28]. To prove our result we use some ideas from that paper.

We use the same notation of Prop. 4.4. We will denote by

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & Z \\ p \downarrow & & \\ \bar{\mathcal{M}}_{0,0}(Z, a\beta) & & \end{array}$$

the universal stable map and by $g : \mathcal{C} \rightarrow E$ the morphism such that $f = j \circ g$ (Lem. 7.1).

Theorem 7.6. *Conjecture 5.1 holds for Z being the crepant resolution of a variety with transversal A_1 -singularities.*

We now prove some lemmas at the end of which we will conclude that Th. 7.6 is true.

Lemma 7.7. *The moduli stack $\bar{\mathcal{M}}_{0,0}(Z, a\beta)$ is smooth of dimension $\dim S = 2a - 2$. The virtual fundamental class is given by*

$$[\bar{\mathcal{M}}_{0,0}(Z, a\beta)]^{vir} = c_r(h^1(E^{\bullet\vee})) \cdot [\bar{\mathcal{M}}_{0,0}(Z, a\beta)] \quad (49)$$

where

$$h^1(E^{\bullet\vee}) \cong R^1 p_*(g^* N_{E/Z}) \quad (50)$$

is a vector bundle of rank $r = 2a - 1$.

Proof. The smoothness of $\bar{\mathcal{M}}_{0,0}(Z, a\beta)$ follows from the fact that the fibers of ϕ are smooth (see Lem. 7.1). Indeed they are all isomorphic to $\bar{\mathcal{M}}_{0,0}(\mathbb{P}^1, a\beta)$. The dimension of $\bar{\mathcal{M}}_{0,0}(Z, a\beta)$ is easily computed. Equation (49) follows from [4]. It remains to prove equation (50).

We first show that $R^1p_*(f^*T_Z)$ is a vector bundle of rank $2a - 1$, and

$$R^1p_*(f^*T_Z) \cong R^1p_*(g^*N_{E/Z}). \quad (51)$$

Let $u = [\mu : D \rightarrow Z] \in \bar{\mathcal{M}}_{0,0}(Z, a\beta)(\text{Spec}\mathbb{C})$ be a stable map. Consider the following exact sequence of locally free sheaves on E

$$0 \rightarrow \mu^*T_E \rightarrow \mu^*T_Z|_E \rightarrow \mu^*N_{E/Z} \rightarrow 0.$$

Since $H^1(D, \mu^*T_E) = 0$, we get

$$H^1(D, \mu^*T_Z) \cong H^1(D, \mu^*N_{E/Z})$$

which has dimension $2a - 1$. This shows that the dimension of $H^1(p^{-1}(u), f^*T_Z)|_{p^{-1}(u)}$ is independent from u , hence $R^1p_*(f^*T_Z)$ is locally free of rank $2a - 1$. To prove (51) we apply $R^\bullet p_*$ to the exact sequence

$$0 \rightarrow T_E \rightarrow T_Z|_E \rightarrow N_{E/Z} \rightarrow 0.$$

□

Lemma 7.8. *We have the exact sequence*

$$0 \rightarrow \phi^*(R^1\pi_*N_{E/Z}) \rightarrow R^1p_*(g^*N_{E/Z}) \rightarrow \mathcal{F} \rightarrow 0,$$

where \mathcal{F} is a vector bundle of rank $2a - 2$ whose restriction on each fiber of $\phi^{-1}(p)$ is now described. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{\phi^{-1}(p)} & \xrightarrow{g|} & E_p \\ p| \downarrow & & \downarrow \pi_p \\ \phi^{-1}(p) & \xrightarrow{\phi|} & \{p\} \end{array}$$

where $\mathcal{C}_{\phi^{-1}(p)}$ is the restriction of \mathcal{C} over $\phi^{-1}(p)$, $p|$ (resp. $g|$) is the restriction of p (resp. g) on it, $E_p = \pi^{-1}(p)$ and π_p is the restriction of π . Then, under the identification of E_p with \mathbb{P}^1 , the restriction of \mathcal{F} to $\phi^{-1}(p)$ is

$$R^1p_{|*}(g|_*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

Proof. Since $E \cong \mathbb{P}(F)$, we have the surjective morphism: $\pi^*(F^\vee) \rightarrow \mathcal{O}_F(1)$. Its kernel is $(\wedge^2 \pi^*(F^\vee)) \otimes \mathcal{O}_F(-1)$. So we have the exact sequence

$$0 \rightarrow (\wedge^2 \pi^*(F^\vee)) \otimes \mathcal{O}_F(-1) \rightarrow \pi^*(F^\vee) \rightarrow \mathcal{O}_F(1) \rightarrow 0,$$

which tensorized with $(\pi^* \wedge^2 F \otimes G) \otimes \mathcal{O}_F(-1)$ yields

$$0 \rightarrow N_{E/Z} \rightarrow \pi^*(F \otimes G) \otimes \mathcal{O}_F(-1) \rightarrow \pi^*(R^1\pi_*N_{E/Z}) \rightarrow 0, \quad (52)$$

using Prop. 4.4.

The pull back under g of (52) gives a short exact sequence of vector bundles on \mathcal{C} . Now, applying the functor $R^\bullet p_*$, we have the long exact sequence:

$$\begin{aligned} 0 &\rightarrow p_*g^*N_{E/Z} \rightarrow p_*(p^*\phi^*(F \otimes G) \otimes g^*\mathcal{O}_F(-1)) \rightarrow p_*p^*\phi^*R^1\pi_*N_{E/Z} \\ &\rightarrow R^1p_*(g^*N_{E/Z}) \rightarrow R^1p_*(p^*\phi^*(F \otimes G) \otimes g^*\mathcal{O}_F(-1)) \\ &\rightarrow R^1p_*(p^*\phi^*R^1\pi_*N_{E/Z}) \rightarrow 0. \end{aligned}$$

Notice that

$$p_*(p^*\phi^*(F \otimes G) \otimes g^*\mathcal{O}_F(-1)) \cong 0$$

by Cohomology and Base Change. Moreover, the projection formula gives

$$\begin{aligned} p_*p^*\phi^*R^1\pi_*N_{E/Z} &\cong \phi^*(R^1\pi_*N_{E/Z}) \\ R^1p_*(p^*\phi^*R^1\pi_*N_{E/Z}) &\cong \phi^*(R^1\pi_*N_{E/Z}) \otimes R^1p_*\mathcal{O}_C \cong 0. \end{aligned}$$

So the thesis follows once defined

$$\mathcal{F} := R^1 p_* (p^* \phi^* (F \otimes G) \otimes g^* \mathcal{O}_F(-1)).$$

□

Theorem 7.6 is now a consequence of the formula

$$\int_{\phi^{-1}(p)} c_{2a-2} (R^1 p_{|*} (g_!^* (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)))) = \frac{1}{a^3}$$

(see e.g. [13]).

7.2 Proof of Conjecture 5.1 in the A_n -case and $\Gamma = \beta_{\mu\nu}$

Theorem 7.9. *Conjecture 5.1 holds for Z the crepant resolution of a variety with transversal A_n singularities such that the associated orbifold has trivial monodromy and $\Gamma = \beta_{\mu\nu}$.*

As in the previous case we need some further results in order to prove the Theorem.

The moduli stack $\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})$ is smooth and isomorphic to S through ϕ (Lem. 7.1). Hence we identify $\bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu})$ with S , so the universal stable map will be

$$\begin{array}{ccc} E & \xrightarrow{j} & Z \\ \pi \downarrow & & \\ \bar{\mathcal{M}}_{0,0}(Z, \beta_{\mu\nu}). & & \end{array}$$

The virtual dimension is $\dim(S) - 1$ and the virtual fundamental class is given by

$$[\bar{\mathcal{M}}_{0,0}(Z, \beta_{ij})]^{vir} = c_1(h^1(E^{\bullet\vee})) \cdot [\bar{\mathcal{M}}_{0,0}(Z, \beta_{ij})]$$

where E^\bullet is the complex [4]

$$E^\bullet = R^\bullet \pi_* ([j^* \Omega_Z \rightarrow \Omega_\pi] \otimes \omega_\pi).$$

Without loss of generality we assume that $\mu = 1$ and $\nu = n$.

Lemma 7.10.

$$h^1(E^{\bullet\vee}) \cong R^1 \pi_* N_{E/Z}.$$

Proof. The complex of sheaves

$$j^* \Omega_Z \rightarrow \Omega_\pi \tag{53}$$

is isomorphic, in the derived category $D(\mathcal{O}_E)$, to a locally free sheaf G in degree -1 . Indeed, the morphism $j^* \Omega_Z \rightarrow \Omega_\pi$ is surjective and, if G denotes its kernel, we have the exact sequence

$$0 \rightarrow G \rightarrow j^* \Omega_Z \rightarrow \Omega_\pi \rightarrow 0.$$

Since Ω_π is of projective dimension one, it follows that G is locally free. So,

$$h^1(E^{\bullet\vee}) \cong R^1 \pi_* (G^\vee).$$

We have the exact sequence

$$0 \rightarrow j^* \mathcal{O}_Z(-E) \rightarrow G \rightarrow \pi^* \Omega_S \rightarrow 0. \tag{54}$$

This follows from a diagram chasing in the next diagram,

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & & \pi^* \Omega_S & & \\
& & & & \downarrow & & \\
0 & \longrightarrow & j^* \mathcal{O}_Z(-E) & \longrightarrow & j^* \Omega_Z & \longrightarrow & \Omega_E \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & = & & \\
0 & \longrightarrow & G & \longrightarrow & j^* \Omega_Z & \longrightarrow & \Omega_\pi \longrightarrow 0 \\
& & & & & & \downarrow \\
& & & & & & 0
\end{array}$$

Then, taking the dual of (54) and applying the functor $R^\bullet \pi_*$ we get the isomorphism

$$R^1 \pi_*(G^\vee) \cong R^1 \pi_* N_{E/Z},$$

which completes the proof. \square

7.3 Proof of Conj. 5.1 for $n \geq 2$ and Γ general

We use the fact that Gromov-Witten invariants are invariant under deformation of the complex structure of Z , so we will add the assumption that some first order deformations of Z are not obstructed.

Notation 7.11. For any variety X , we will denote by T_X the sheaf of \mathbb{C} -derivations, i.e.,

$$T_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X),$$

where Ω_X^1 is the sheaf of differentials of X .

Assume that $H^2(Z, T_Z) = 0$, then we have the following exact sequence of cohomology groups

$$0 \rightarrow H^1(Y, T_Y) \rightarrow H^1(Z, T_Z) \rightarrow H^0(Y, R^1 \rho_* T_Z) \rightarrow 0. \quad (55)$$

This is the Leray spectral sequence associated to the morphism ρ .

Remark 7.12. $H^1(Y, T_Y)$ is in 1-1 correspondence with the set of equivalence classes of first order deformations of Y which are locally trivial. On the other hand $H^1(Z, T_Z)$ is in 1-1 correspondence with the set of equivalence classes of first order deformations of Z modulo isomorphisms. Therefore the sequence (55) has the following meaning in deformation theory: to any first order locally trivial deformation of Y we can associate a first order deformation of Z , the remaining deformations of Z come from $H^0(Y, R^1 \rho_* T_Z)$. We are interested in understanding the last deformations.

Lemma 7.13. *Let*

$$\text{ob}_{\mu\nu} : R^1 \rho_* T_Z \rightarrow i_*(R^1(\pi_{\mu\nu})_* N_{E_{\mu\nu}/Z}) \quad (56)$$

be the composition of $R^1 \rho_ T_Z \rightarrow i_*(R^1 \pi_*(j^*(T_Z)))$ with $i_*(R^1 \pi_*(j^*(T_Z))) \rightarrow i_*(R^1(\pi_{\mu\nu})_* N_{E_{\mu\nu}/Z})$. Then $\text{ob}_{\mu\nu}$ is surjective and, if $\mathcal{D}_{\mu\nu}$ denotes its kernel, we have the following exact sequence*

$$0 \rightarrow \mathcal{D}_{\mu\nu} \rightarrow R^1 \rho_* T_Z \xrightarrow{\text{ob}_{\mu\nu}} i_*(R^1(\pi_{\mu\nu})_* N_{E_{\mu\nu}/Z}) \rightarrow 0. \quad (57)$$

Proof. Over an open set $W \subset Y$ isomorphic to $\mathbb{C}^k \times R$, we have

$$H^0(W, R^1 \rho_* T_Z) \cong H^1(\mathbb{C}^k \times \tilde{R}, T_{\mathbb{C}^k \times \tilde{R}}) \cong H^0(\mathbb{C}^k, \mathcal{O}_{\mathbb{C}^k}) \otimes H^1(\tilde{R}, T_{\tilde{R}}).$$

This follows from Künneth formula and the fact that the surface singularity is rational. On the other hand, let $C = C_1 + \dots + C_n$ be the exceptional divisor of $\tilde{R} \rightarrow R$ and $C_{\mu\nu} = C_\mu + \dots + C_\nu$, then

$$H^0(W, i_*(R^1(\pi_{\mu\nu})_* N_{E_{\mu\nu}/Z})) \cong H^0(\mathbb{C}^k, \mathcal{O}_{\mathbb{C}^k}) \otimes H^1(C_{\mu\nu}, N_{C_{\mu\nu}/\tilde{R}}).$$

It is hence enough to show that

$$H^1(\tilde{R}, T_{\tilde{R}}) \rightarrow H^1(C_{\mu\nu}, N_{C_{\mu\nu}/\tilde{R}})$$

is surjective. This follows from the fact that the morphisms $H^1(\tilde{R}, T_{\tilde{R}}) \rightarrow H^1(C_{\mu\nu}, T_{\tilde{R}} \otimes \mathcal{O}_{C_{\mu\nu}})$ and $H^1(C_{\mu\nu}, T_{\tilde{R}} \otimes \mathcal{O}_{C_{\mu\nu}}) \rightarrow H^1(C_{\mu\nu}, N_{C_{\mu\nu}/\tilde{R}})$ are surjective. To prove surjectivity of the second morphism we consider the exact sequence

$$0 \rightarrow T_{C_{\mu\nu}} \rightarrow T_{\tilde{R}} \otimes \mathcal{O}_{C_{\mu\nu}} \rightarrow N_{C_{\mu\nu}/\tilde{R}} \rightarrow \text{Ext}_{\mathcal{O}_{C_{\mu\nu}}}^1(\Omega_{C_{\mu\nu}}, \mathcal{O}_{C_{\mu\nu}}) \rightarrow 0.$$

Let \mathcal{F} denote the image of the morphism $T_{\tilde{R}} \otimes \mathcal{O}_{C_{\mu\nu}} \rightarrow N_{C_{\mu\nu}/\tilde{R}}$, then we get two exact sequences

$$0 \rightarrow T_{C_{\mu\nu}} \rightarrow T_{\tilde{R}} \otimes \mathcal{O}_{C_{\mu\nu}} \rightarrow \mathcal{F} \rightarrow 0,$$

$$0 \rightarrow \mathcal{F} \rightarrow N_{C_{\mu\nu}/\tilde{R}} \rightarrow \text{Ext}_{\mathcal{O}_{C_{\mu\nu}}}^1(\Omega_{C_{\mu\nu}}, \mathcal{O}_{C_{\mu\nu}}) \rightarrow 0.$$

From the long exact sequences of cohomology groups we have that the morphisms $H^1(C_{\mu\nu}, T_{\tilde{R}} \otimes \mathcal{O}_{C_{\mu\nu}}) \rightarrow H^1(C_{\mu\nu}, \mathcal{F})$ and $H^1(C_{\mu\nu}, \mathcal{F}) \rightarrow H^1(C_{\mu\nu}, N_{C_{\mu\nu}/\tilde{R}})$ are surjective. Since $\text{Ext}_{\mathcal{O}_{C_{\mu\nu}}}^1(\Omega_{C_{\mu\nu}}, \mathcal{O}_{C_{\mu\nu}})$ is a sheaf supported on the nodes of $C_{\mu\nu}$. This complete the proof. \square

Remark 7.14. Notice that $R^1\rho_*T_Z$ is locally free of rank n and is supported on S . Since also $R^1(\pi_{\mu\nu})_*N_{E_{\mu\nu}/Z}$ is locally free of rank 1 (Lem. 5.5), it follows that $\mathcal{D}_{\mu\nu}$ is locally free of rank $n-1$ and it is supported on S .

Theorem 7.15. *Let Z be the crepant resolution of a variety with transversal A_n singularities such that the associated orbifold has trivial monodromy. Assume furthermore that $H^2(Z, T_Z) = 0$ and that, for any $\mu, \nu \in \{1, \dots, n\}$ with $\mu \leq \nu$, there exists a global section of $R^1\rho_*T_Z$ which intersects $\mathcal{D}_{\mu\nu}$ transversally (see Lemma 7.13). Then Conjecture 5.1 holds for Z .*

Proof. We first prove that

$$\Psi_{\Gamma}^Z(\gamma_1, \gamma_2, \gamma_3) = 0 \quad \text{if} \quad \Gamma \neq a\beta_{\mu\nu}. \quad (58)$$

Let $\sigma \in H^0(Y, R^1\rho_*T_Z)$ be a section and consider a first order deformation of Z associated to σ (Rem. 7.12):

$$\begin{array}{ccc} Z & \longrightarrow & Z_1 \\ \downarrow & & \downarrow \\ \text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}\left(\frac{\mathbb{C}[\epsilon]}{\epsilon^2}\right). \end{array} \quad (59)$$

There exists a finite deformation of Z which at the first order coincides with (59), we will denote this deformation by

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta \end{array} \quad (60)$$

where Δ is a small disc in \mathbb{C} around the origin $0 \in \mathbb{C}$ (this is the Kodaira-Nirenberg-Spencer Theorem (1958) [25], see also [42] for a review).

We cover a neighborhood of E in Z with open subsets V of the form

$$V \cong U \times \tilde{R}$$

where $U \subset S$ is isomorphic to an open ball in \mathbb{C}^k . Then (60) induces a deformation of V :

$$\begin{array}{ccc} V & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta. \end{array} \quad (61)$$

Let H be the semi-universal deformation space of \tilde{R} and let $\tilde{\mathcal{R}} \rightarrow H$ be the semi-universal family [6] (see also [24] and [45]). H is an n -dimensional complex vector space and there is a natural isomorphism

$$H^1(\tilde{R}, T_{\tilde{R}}) \cong T_{H,0}.$$

Then under the identification

$$(R^1\rho_*T_Z)|_U \cong H^0(U, \mathcal{O}_U) \otimes H^1(\tilde{R}, T_{\tilde{R}})$$

the restriction of σ to U corresponds to a function

$$\sigma_U : U \rightarrow H^1(\tilde{R}, T_{\tilde{R}}) \cong T_{H,0}.$$

Moreover (61) can be obtained as a pull-back of the semi-universal family $\tilde{\mathcal{R}} \rightarrow H$ under a morphism

$$\Sigma_U : U \times \Delta \rightarrow H$$

such that

$$\frac{\partial}{\partial t}|_{t=0} \Sigma_U = \sigma_U$$

where t denotes the variable in Δ .

A generic deformation of \tilde{R} has no complete curves. More precisely, the discriminant locus $D \subset H$ is defined as the set of points $h \in H$ such that the corresponding surface $\tilde{\mathcal{R}}_h$ has a complete curve (see [8] Prop. 2.2). It turns out that

$$D = \cup_{\mu \leq \nu} D_{\mu\nu}$$

where $D_{\mu\nu}$ are hyper-planes through the origin (see [6], [8], [24], [45]). Moreover a generic point of $D_{\mu\nu}$ corresponds to a deformation of \tilde{R} with a complete curve whose homology class is $[C_\mu] + \dots + [C_\nu]$ (where $C = C_1 + \dots + C_n$ is the exceptional divisor of $\rho : \tilde{R} \rightarrow R$). Thus the locus of points $p \in U$ where the curve $\pi_{\mu\nu}^{-1}(p)$ deforms in \mathcal{V} has codimension 1, and the locus of points where curves in the fibers of π of different homology classes deforms has codimension greater than 1.

Since the expected dimension of $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)$ is $\dim(S) - 1$, (58) follows (see (47)).

Now we consider the case $\Gamma = a\beta_{\mu\nu}$. From (47) it is enough to prove that

$$\int_{[\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{\text{vir}}} \phi^*(\alpha_1 \cdot \alpha_2 \cdot \alpha_3) = \frac{1}{a^3} \int_S \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot R^1 \pi_* N_{E/Z}. \quad (62)$$

To get this result we construct a deformation of Z ,

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta, \end{array} \quad (63)$$

in the following way. We choose a section $\sigma \in H^0(Y, R^1 \rho_* T_Z)$ which intersects transversally $\mathcal{D}_{\mu\nu}$. Then (63) is given by σ in the same way as (60). The deformation invariance property implies

$$\int_{[\bar{\mathcal{M}}_{0,0}(Z, \Gamma)]^{\text{vir}}} \phi^*(\alpha_1 \cdot \alpha_2 \cdot \alpha_3) = \int_{[\bar{\mathcal{M}}_{0,0}(\mathcal{Z}_t, \Gamma)]^{\text{vir}}} \phi^*(\alpha_1 \cdot \alpha_2 \cdot \alpha_3)$$

for any $t \in \Delta$. We claim that for generic $t \in \Delta$ the number of rational curves in \mathcal{Z}_t that pass through three sub-varieties of class $\alpha_1 E_{l_1}$, $\alpha_2 E_{l_2}$ and $\alpha_3 E_{l_3}$ is

$$[\{p \in S : \text{ob}_{\mu\nu}(\sigma) = 0\}] \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

and moreover, each of this curve is isomorphic to \mathbb{P}^1 with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus \dim S}$. Then the result follows from the fact that the Poincaré dual of the homology class $[\{p \in S : \text{ob}_{\mu\nu}(\sigma) = 0\}]$ is $c_1(K)$ (Lem. 5.5) and from the Aspinwall-Morrison formula (see e.g. [13] Th. 7.4.4).

We first show that for a generic $t \in \Delta$ the locus in \mathcal{Z}_t of complete curves of homology class Γ is homologous to

$$\pi_{\mu\nu}^{-1}(\{p \in S : \text{ob}_{\mu\nu}(\sigma) = 0\}) \subset Z. \quad (64)$$

First notice that the set (64) is the locus of curves in Z of homology class Γ which deform in \mathcal{Z}_1 (the first order deformation of Z induced by (63)). Indeed fiber-wise the morphism $\text{ob}_{\mu\nu}$ is the morphism

$$H^1(\tilde{R}, T_{\tilde{R}}) \rightarrow H^1(C_{\mu\nu}, N_{C_{\mu\nu}/\tilde{R}})$$

which associates to any first order deformation of \tilde{R} the obstruction to extend $C_{\mu\nu}$ in such deformation [42]. Then, we notice that, if a curve $\pi_{\mu\nu}^{-1}(p)$ deforms in \mathcal{Z} at the first order, then it deforms in \mathcal{Z} for generic $t \in \Delta$. To see this, let $U \subset S$ open neighborhood of $p \in S$ isomorphic to a ball in \mathbb{C}^k with coordinates (x_1, \dots, x_k) . The deformation \mathcal{Z} induces a deformation \mathcal{V} as (60). As before \mathcal{V} is given by a holomorphic map

$$\Sigma_U : U \times \Delta \rightarrow H$$

such that

$$\frac{\partial}{\partial t}|_{t=0} \Sigma_U = \sigma_U.$$

Our hypothesis imply that

$$\begin{aligned}\Sigma_U(x, 0) &= 0 \quad \text{for all } x \in U, \\ \frac{\partial}{\partial t}|_{(p, t=0)} \Sigma_U &\in D_{\mu\nu}, \\ \frac{\partial}{\partial x_l} \frac{\partial}{\partial t}|_{(p, t=0)} \Sigma_U &\notin D_{\mu\nu} \quad \text{for some } l.\end{aligned}$$

Using the first condition we have $\Sigma_U(x, t) = t\tilde{\Sigma}_U(x, t)$ for some function $\tilde{\Sigma}_U : U \times \Delta \rightarrow H$. Then the claim follows from the implicit function theorem applied to $\tilde{\Sigma}_U(x, t)$. To compute the normal bundle of these rational curves, notice that locally \mathcal{Z}_t is isomorphic to the product of \mathbb{C}^{k-1} ($k = \dim S$) with the semi-universal deformation of the resolution of the A_1 -singularity. \square

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